The Dissertation Committee for Qimou Su
certifies that this is the approved version of the following dissertation:

Essays on Derivatives Pricing in Incomplete Financial Markets

Committee:

Thaleia Zariphopoulou, Supervisor

Luis A. Caffarelli

Oscar Gonzalez

Ehud I. Ronn

Stathis Tompaidis

Essays on Derivatives Pricing in Incomplete Financial Markets

by

Qimou Su, M.S., B.S.

Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

December 2007

To my family

Acknowledgments

I would like to express my sincere gratitude and appreciation to my supervisor, Dr. Thaleia Zariphopoulou, for her insightful guidance and substantial advice throughout this research. I, in particular, appreciate her for introducing me into this wonderful area, inspiring my numerous research ideas, and granting me generous financial support.

I am truly grateful to all the members of my committee, who spent their precious time in reading this dissertation and offered valuable advice during my graduate career at UT-Austin. Together with other professors, they helped me build up a solid foundation in the areas of Applicable Mathematics, Scientific Computation, Economics and Finance, and paved the path for my further exploration in Quantitative Finance.

In particular, I appreciate Dr. Luis Caffarelli and Dr. Panagiotis Souganidis for inspiring my interest in Applied Mathematics and PDEs. I owe my knowledge in the areas of Probability, Stochastic Processes, Stochastic Calculus, SDEs, Stochastic Control, and their applications in finance to Dr. Ari Arapostathis, Dr. Jack Xin, Dr. Thaleia Zariphopoulou, and Dr. Gordan Žitković. The knowledge that I gained from them contributes to the core of my research.

I express my gratitude to Dr. Stathis Tompaidis for introducing me to various computational techniques in finance and for his valuable suggestions on my research and career in quantitative finance. My heartfelt appreciation goes to Dr.

Oscar Gonzalez, Dr. Alan Cline and Dr. Inderjit Dhillion, who assisted me in the areas of Numerical Linear Algebra, Numerical Analysis, and Numerical Differential Equations. Thank you to Dr. David Morton for introducing me to Stochastic Optimization and Monte Carlo methods, which are very helpful for my research.

I am very thankful to Dr. Ehud Ronn for his helpful comments and suggestions from the perspective of finance. I deeply appreciate Dr. Hong Yan for his initiation of my study in Asset Pricing and his sincere suggestions on my career development. Thank you to Dr. Svetlana Boyarchenko and Dr. Andres Almazan whose excellent teaching in Economics and Corporate Finance inspired me to refine my research ideas. My special gratitude goes to Dr. Curt Randall for providing me the opportunity to learn the latest industrial applications of derivatives pricing and risk management and inspiring me to improve my quantitative and real-world problem solving skills.

I would like to gratefully acknowledge the financial support from the Institute for Computational Engineering and Sciences (ICES) and the Department of Mathematics. Their generous support makes the completion of this research possible. I am especially thankful to Dr. Todd Arbogast, Dr. William Beckner, Dr. Clint Dawson, Dr. Bruce Palka, Dr. Lorenzo Sadun and Dr. Jack Xin for their generous advice towards my graduate study in Computational and Applied Mathematics (CAM) and my teaching career in Mathematics. I wish to thank all the members of the administrative staff in ICES and Mathematics for creating such a friendly and comfortable working environment. Many thanks in particular to Nancy Lamm for her friendship, kindness and numerous suggestions.

Finally, I thank my parents and family for their years of dedication and support and for their sincere concern about my life in this foreign land. My deepest appreciation goes to Ni, my dear wife, for her love, support and encouragement all along.

Qimou Su

The University of Texas at Austin
December 2007

Essays on Derivatives Pricing in Incomplete Financial Markets

Publication No. ______________________

Qimou Su, Ph.D.
The University of Texas at Austin, 2007

Supervisor: Thaleia Zariphopoulou

This dissertation is a contribution to the valuation and risk management of derivative securities in incomplete financial markets. It consists of two parts dedicated to two distinct valuation methodologies.

In the first part, we develop a valuation approach based on equilibrium arguments from the perspective of option market makers and financial intermediaries. This approach produces a new pricing concept that we call the *competition-based price*. We analyze such prices in both a semimartingale and a diffusion setting. The emerging pricing measure is characterized as the *minimal entropy martingale measure* (MEMM) with respect to a *new prior*. This new prior depends on the aggregate demand and inventory of the derivatives and is characterized as an *Esscher*

transform of the historical measure. In a diffusion setting, the pricing measure is explicitly constructed. We show that the competitive price of a derivative is an increasing function of the demand of any derivative in the market. The increasing rate is proportional to the covariance between the unhedgeable parts of the associated derivative payoffs, calculated under the competition-based pricing measure. This result may contribute to the resolution of some of the well known *option-pricing puzzles*. We further compare our approach to existing pricing methodologies, such as the *marginal-utility pricing* and *indifference valuation*. In addition, we apply our approach to price a family of volatility derivatives. We develop numerical schemes based on Monte Carlo simulations for a Heston-type stochastic volatility model.

In the second part, we apply the well established indifference approach to value options with staging structure and sequential decisions, such as *installment options* and *venture capital* contracts. In a diffusion market setting, we analyze the underlying stochastic optimization problems via the associated Hamilton-Jacobi-Bellman equations. We deduce a quasilinear PDE for the indifference price and analyze it probabilistically. We also obtain an explicit pricing formula under appropriate market restrictions and characterize the indifference price as a *nonlinear expectation* under the MEMM. The associated hedging and risk monitoring strategies are investigated. We further develop numerical schemes based on regression techniques to value the ASX Installments and the staged financing of venture capital. Moreover, a *foresighted valuation* framework is introduced to incorporate the investors' private information into their valuation and hedging strategies. Such information may include both their *ex-ante* risk exposure and *ex-post* investment opportunities. Finally, we adopt the recently developed *dynamic performance criteria* to price volatility derivatives. We develop numerical schemes for the computation of the *forward and backward indifference prices* in models of Heston and reciprocal-Heston type.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

> "When banks compete, you win." – *An advertising logo of LendingTree, LLC*

> "If people are willing to pay foolish money for insurance, why shouldn't we sell it to them?" – *Lowenstein (2000)*

> "Current asset pricing theories usually assume investors choose optimal portfolios directly. The fact that there is such extensive intermediation suggests that this approach may miss important features of actual markets." – *Allen and Santomero (1998)*

> "I believe a renewed focus on the explicit financial intermediation of the underlying risks by option market markers is needed. [...] Devising plausible models of market maker behavior under more general risks and incorporating them into equilibrium models of risk pricing is desirable." – *Bates (2003)*

Over thirty years after its discovery, the Black-Scholes option pricing theory is still the only universally accepted methodology in pricing derivatives [see, among others, 14, 97]. In the Black-Scholes-Merton framework, models are developed based on the fundamental assumption that financial market is complete. The derivatives are assumed to be redundant securities, whose payoffs can be perfectly replicated through dynamic trading in the underlying securities. The derivative price is then defined as the value of the replicating portfolio according to the no-arbitrage principle. This assumption, however, fails to justify the existence of derivatives in the real world, as argued by [66]. In reality, many derivatives are introduced to the markets when they are useful to manage various intrinsic risks, such as credit, mortality, volatility, and weather risks. Once such non-marketable risks are involved, not all derivatives can be replicated. This kind of markets are called the incomplete financial markets. Clearly, the Black-Scholes theory breaks down in this situation. The main difficulty here is that there are infinite equivalent martingale measures. The no-arbitrage assumption is no longer sufficient to determine the pricing measure and thus the derivative price. Many methods are studied to specify the appropriate pricing concept and the associated pricing measure. However, no consensus has been achieved.

In this dissertation, we attempt to tackle this difficulty by replacing the no-arbitrage arguments by the investment optimality of the market players and their financial interactions. In the first part, we develop a valuation approach based on equilibrium arguments to price derivatives from the perspective of option market makers and financial intermediaries. In the second part, we apply the well established indifference approach to value derivatives with staging structure such as installment options and venture capital contracts. We also adopt the dynamic performance criteria developed recently by [113, 114] to price a family of volatility derivatives. The remainder of the dissertation consists of five relative independent chapters, which are organized as follows.

In Chapter 2, we model the competition among an arbitrary number of market makers in an incomplete financial market containing multiple non-redundant derivatives. This approach produces a new pricing concept that we call the *competition based price*. In a Markovian diffusion setting, we derive a closed-form formula for the prices of derivatives. The emerging pricing measure is characterized as an *Esscher transform* of the *minimal entropy martingale measure* (MEMM). We further establish the various properties of the competition-based price including the price effects of demand and inventory pressure. We find that the competition-based price of a derivative is an increasing function of the demand of any derivative in the market. Moreover, the increasing rate is proportional to the covariance between the unhedgeable parts of the payoffs of the associated derivatives, calculated under the competition-based pricing measure. This result contributes to the resolution of some of the well known *option-pricing puzzles*. We further compare our approach to existing pricing methodologies, such as *marginal utility pricing* and *indifference valuation*. In addition, we apply our model to price a family of volatility derivatives. We develop the numerical schemes based on Monte Carlo simulations for a Heston-type stochastic volatility model.

In Chapter 3 , we extend the equilibrium approach to price derivatives from the viewpoint of financial intermediation. We consider an incomplete semimartingale market that consists of n fundamental securities and an arbitrary number of non-redundant derivatives. We adopt Merton's (1993) *functional perspective* to model the optimal behavior of intermediation. Our model allows an arbitrary number of intermediaries in the market. Using the convex duality techniques, we analyze the utility maximization problems of the financial intermediaries. We further establish the competitive equilibrium in the derivative market, which produces the competition-based prices for the derivatives. The emerging pricing measure turns out to be the MEMM with respect to a *new prior*. This new prior depends on the aggregate demand and inventory of the derivatives and is characterized as an *Esscher transform* of the historical measure. A sensitivity analysis on the price effects of such demand and inventory pressure further confirms that the price of a derivative is increasing with the demand of any derivative in the market. Moreover, the increasing rate is proportional to the covariance between the unhedgeable parts of the payoffs of the associated derivatives, calculated under the competition-based pricing measure. We, finally, provide examples in the Markovian framework such as stochastic volatility model.

In Chapter 4, we study the well established indifference valuation in a Markovian diffusion setting. This valuation methodology is applied to value a portfolio of n derivatives with distinct maturities. We analyze the underlying stochastic optimization problems via the associated Hamilton-Jacobi-Bellman equations. We deduce a quasilinear PDE for the indifference price and analyze its probabilistic representations using the Feynman-Kac connection. Moreover, we also obtain an explicit pricing formula under appropriate market restrictions and characterize the indifference price as a *nonlinear expectation* under the MEMM. Properties of the pricing functionals are further established based on the pricing PDEs. In addition, the risk monitoring strategies and payoff decompositions are developed and compared with their counterparts in the Black-Scholes model. Finally, a *foresighted valuation* framework is introduced to value irreversible investments that may involve staging structure and sequential decisions. This framework emphasizes on incorporating investors' private information into their valuation and hedging strategies. Such information includes not only their *ex-ante* risk exposure but also their *ex-post* investment opportunities such as "contingent pre-contracting" and "staged financing" in venture capital investments.

In Chapter 5, we apply the indifference approach to value *installment options* in incomplete markets. An installment option differs from a conventional option because its premium is paid in installments (scheduled periodically over the life of the option) instead of a lump sum upfront. Also, the holder has the right to terminate the contract by defaulting any installment payment, which leaves no further liability to either party. The installment option to be priced herein has a payoff depending on both tradable securities and non-traded risk factors. For a certain class of models, the pricing formula and the associated optimal abandoning boundaries are obtained in a closed form. The dynamic hedging and risk monitoring strategies are further investigated. In addition, we develop numerical schemes based on regression techniques and illustrate the applications of the model in the analysis of installment warrants and staged financing of venture capital.

In Chapter 6, we present a utility approach to value volatility derivatives based on the *dynamic performance criteria*. The dynamic performance criteria are valuation tools recently developed by [113, 114]. In a Markovian setting, we apply a forward performance criterion of exponential type to construct the *forward indifference price*. We further obtain a closed-form pricing formula. Different from the traditional *backward indifference price*, this forward indifference price is characterized as a nonlinear expectation under the *minimal martingale measure* (MMM). We further investigate the associated optimal portfolio and risk monitoring strategy. Numerical procedures based on the Laplace transform and Monte-Carlo simulations are developed for the computation of the forward and backward indifference prices. Finally, we illustrate their applications to value a family of volatility derivatives in models of Heston and reciprocal-Heston type.

Part I

Equilibrium Approach

Chapter 2

Derivatives Pricing from the Perspective of Market Makers

2.1 Introduction

This paper presents an equilibrium approach to price non-redundant derivatives from the perspective of options market makers (dealers). We model the competition among an arbitrary number of market makers in an incomplete dealership market, and analyze the impact of the derivative demand and inventory pressure on the valuation of derivatives. Our aim is twofold, namely: (i) to expand the economic insights of the *utility-based pricing* mechanism, and (ii) to provide a possible resolution for the so called *option-pricing puzzles*.

In the classical Black-Scholes-Merton framework, derivatives are priced based on the assumption that they are redundant securities, whose payoffs can be perfectly replicated through dynamic trading in the underlying securities [see, among others, 14, 97]. Such an assumption of redundancy, together with the no-arbitrage principle, enable us to deduce the unique ("preference-less") derivative price from the value of the associated replicating portfolio. This assumption, however, fails to justify the existence of derivatives in the real world, as argued by [66]. Many derivatives are introduced to the markets when they are useful to manage various intrinsic risks, such as credit, mortality, volatility, and weather risks. Once such non-marketable risks are involved, the Black-Scholes options pricing theory breaks down because these risks are in general unhedgeable. The main difficulty lies in the fact that, in such an incomplete market, there are infinite equivalent martingale measures.

One remedy of this difficulty is to replace the no-arbitrage arguments by the investment optimality of the market players (as well as their financial interactions). The starting point is that the traders' risk preference matters in making their trading decisions on derivatives when they are not able to completely hedge the associated risks. From this angle, the trader's risk preferences, usually quantified by a utility function, are essential in derivative pricing. This motivates the development of utility-based pricing methodology, which recently has become a rapidly growing body of research in the literature.

The first strand of literature proposes the notion of *indifference price*, which is defined as the amount such that the representative trader is indifferent – in terms of optimal expected utility – between trading and not trading the derivative. This valuation concept was introduced to price European options by [72] and further extended by [35] in dynamic settings, though its static analogue, the *certainty-equivalent price*, was studied earlier in the actuarial science [see 17, 59]. By now, the indifference prices have been studied extensively in quantitative finance; see, among others, [118, 40, 111], and [83].

Worth noting is that the indifference prices are essentially the trader's reservation prices, defining a range in which trading can take place. Specifically, the indifference buying price is the highest one that the buyer is willing to buy the derivative, while the selling price is the lowest one that the seller is willing to sell. Due to the nonlinear nature of the indifference pricing mechanism, the buying price is, in general, different with the selling one for a nontrivial claim. Moreover, as shown by [129], the selling price is always higher than the buying one, which implies that such a pricing criterion generally does not allow an equilibrium in the derivative market [see also 35].

Alternative to the indifference approach, [32] proposed the notion of *marginal utility price* ("fair price") based on the idea of "zero marginal rate of substitution". Precisely, the marginal price is defined to be such that the representative agent cannot increase his/her optimal expected utility by diverting an infinitesimal amount of capital into the derivative account. Unlike the indifference prices, the marginal price is linear, symmetric for buyer and seller, and it turns out to be the asymptotic limit of the average (per unit) indifference price for an infinitesimal quantity of derivatives. This pricing concept has also been studied by, among others, [79, 80] and [51]. The uniqueness of the marginal price was proved by [74]. More recently, [85, 86] analyzed its sensitivity and asymptotic property.

To the best of our knowledge, most of these models are based on the assumption that there is a single representative trader (the market maker) who has the power to determine the derivative prices. In reality, however, market makers, although having the pricing power, are subject to competing with each other when providing liquidity service for their customers (impatient traders). This kind of competition is in fact a common feature in many exchange markets, such as New York Stock Exchange (NYSE), NASDAQ, and London Stock Exchange. For example, as noted by [134], market makers such as specialists in NYSE typically face competition from floor traders, competing dealers, limit orders, and other exchanges.[1] In market microstructure models, the competition among market makers has been studied early by [70] and others.

Also commonly assumed in the literature is that the derivatives have zero demand and inventory pressure, which presumably erases their effects on the prices. This is, however, inconsistent with what is observed in the real world. In fact, market makers actively adjust their price quotation in response to the fluctuations in the public demand pressure and their own inventory level [see 130, 57].[2] Moreover, recent empirical research has shown a strong correlation between the derivative prices and the demand (buying) pressure [see, among others, 15, 21, 22]. Based on such intensive evidence, we believe that the demand and inventory pressure should play a significant role in determining the derivatives prices.

In this paper, we build a competition-based derivative pricing model that takes into account the impact of demand and inventory. In contrast to the utility-based framework, we allow an arbitrary number of risk-averse market makers, who compete to supply the derivatives for the demand from their customers (impatient traders). Because these derivatives involve unhedgeable risks, the market makers control their net inventories of risk, and subsequently, create dynamic trading strate-

[1] Limit orders are orders to buy (sell) that specify a maximum (minimum) price at which the trader is willing to transact. The traders who issue limit orders are indeed acting as dealers, though they often do not recognize this. A market order is an order to buy (sell) at prevailing prices. The traders that submits market order can be viewed as customers. For more discussion, we refer to [67].

[2] Market makers usually have their own target levels of inventory. When there is an order imbalance that moves market makers away from their desired inventory position, they adjust their price quotation to attract orders to move back to their target inventory positions [see 67, Chapter 13].

gies to hedge part of their net exposure. Managing these two activities optimally – in terms of utility maximization – they are able to choose their optimal derivative allocations and hedging strategies for any given security prices. When the market is in equilibrium, the derivative prices must clear the market. This market clearing condition is, in turn, applied to deduce the equilibrium prices of the derivatives.

In a Markovian diffusion setting, we establish the competitive equilibrium and derive a closed-form formula for the competitive prices of the derivatives. The associated pricing measure (pricing kernel) can be characterized as an *Esscher transform* [see 60] of the *minimal entropy martingale measure*. The Esscher transform herein involves the level of aggregate demand and inventory, which can be identified directly from the market data [see, for example, 15]. Further, it turns out that, when ignoring the impact of demand and inventory, the competition-based price reduces to the Davis' marginal price. From this perspective, the concept of competition-based price generalizes that of the marginal price. Moreover, our model provides an economic framework for understanding the utility-based pricing mechanism and bridges the latter with the classical asset equilibrium concepts.

Besides the above modelling advantage, our work may contribute to the resolution of the well-known *option-pricing puzzles*, namely, index options appear to be expensive and low-moneyness options seem to be especially expensive comparing to other individual equity options. These puzzles have been well documented in the literature of empirical options pricing; see, among others, [91, 9, 28, 16, 5], and [26].

Our model shows that the price of a derivative is increasing with the demand of any derivative in the derivative market. The increasing rate is proportional to the covariance between the unhedgeable parts of the associated derivative payoffs, calculated under the competitive pricing measure. This result helps to explain the options-pricing puzzles because it is evident that the demands of index options and deep out-of-the-money (OTM) puts are very high comparing to other individual equity options; see, among others, [15, 21, 22, 56]. In addition, our work relates to the optimal positioning in derivatives [see, among others, 20, 19, 90, 76, 77]. Another stream of literature related to our approach is the market completion [see, for example, 73, 23].

The remainder of this paper is organized as follows. In Section 2.2, we describe the competition-based pricing model in a Markovian diffusion setting. In Section 2.3, we derive the competition-based pricing formula and analyze the price effects (sensitivity) of the demand and inventory pressure. In Section 2.4, we investigate the various properties of competitive price as well as its relations to some existing pricing concepts. In Section 2.5, we illustrate an application to volatility derivatives and further develop the associated numerical procedures for Heston stochastic volatility model. Finally, we conclude in Section 2.6.

2.2 A Competitive Equilibrium Model

2.2.1 The Market

We fix a finite horizon $T < \infty$ and consider a continuous-time financial market consisting of a riskless bond B (money-market account) and a risky stock S, whose prices are given exogenously. The stock price is assumed to follow a diffusion process satisfying

$$dS_t = \mu(t, Y_t)S_t dt + \sigma(t, Y_t)S_t dW_t^1, \quad t \geq 0, \tag{2.1}$$

with $S_0 > 0$. The drift μ and volatility σ of the stock are driven by a stochastic factor Y, which is modelled as a correlated diffusion

$$dY_t = b(t, Y_t)dt + a(t, Y_t)(\rho dW_t^1 + \bar{\rho} dW_t^{\perp}), \quad t \geq 0, \tag{2.2}$$

with $\rho \in (-1, 1)$ being the correlation coefficient and $\bar{\rho} = \sqrt{1-\rho^2}$. The bond is assumed to mature at T and be tradable over the time horizon $[0, T]$, yielding constant interest rate r. Without loss of generality we take $r = 0$, which is equivalent to use the bond as numeraire. The results for $r > 0$ follow directly from the standard rescaling arguments.

In (2.1) and (2.2), the processes $(W_t^1, W_t^{\perp}; t \geq 0)$ are independent standard Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the P-augmented filtration generated by $(W^1, W^{\perp})$. The market coefficients μ, σ, a, and b are assumed to satisfy all the regularity conditions such that equations (2.1) and (2.2) have a unique strong solution satisfying $S_s > 0$ P-a.s. for a.e. $s \in [t, T]$. The *Sharpe ratio* process of the stock is defined by $\lambda_t = \lambda(t, S_t), t \geq 0$ with $\lambda(t, S) := \mu(t, S)/\sigma(t, S)$.

We denote by (π, x) a self-financing portfolio with initial capital x and π

being the amount invested in the stock. Then, direct calculation shows that the corresponding *wealth* process $X^{\pi,x}$ satisfies the following controlled diffusion equation

$$dX_s = \mu(s, Y_s)\pi_s ds + \sigma(s, Y_s)\pi_s dW_s^1, \quad t \le s \le T, \tag{2.3}$$

with $X_t = x \in \mathbb{R}$ [see 96]. A control process π is called admissible if it is $\mathcal{F}_s$-predictable and satisfies the integrability condition $E\int_0^T \sigma^2(s, Y_s)\pi_s^2 ds < +\infty$. We denote by $\mathcal{A}$ the set of admissible trading strategies.

Besides the fundamental (underlying) securities, there are also n ($n \ge 1$) derivatives available in the market, which are characterized in terms of their terminal payoff at maturity T. The derivative payoffs are given exogenously by n $\mathcal{F}_T$-measurable random variables $(G^1, \cdots, G^n)$. To avoid triviality, we assume that all these derivatives are non-redundant in the sense that any linear combination of $(G^1, \cdots, G^n)$ is not replicable by an admissible trading strategy in the underlyings, namely

$$\exists\, \pi \in \mathcal{A} \quad X_T^{\pi,x} + \alpha \cdot G = \text{const} \quad P\text{-a.s.} \quad \Longrightarrow \quad \alpha = 0. \tag{2.4}$$

In this paper we are interested in the initial price at time t, $p_t := (p_t^1, \cdots, p_t^n)$, of the derivatives $G := (G^1, \cdots, G^n)$. For technical reasons arising from the underlying utility maximization, we assume throughout that G is bounded. Although some derivatives have an unbounded payoff, it will not affect the practical accuracy to truncate the payoff conditional on sufficiently extreme events. For discussion on how to relax this assumption, we refer the reader to [10].

2.2.2 The Traders

In our derivative market, there are two kinds of traders – *market makers* ("dealers") and their *customers* ("impatient traders"). The impatient traders are those who trade derivatives for some inherent reasons. They either want to satisfy some internal requirements such as agency problem, risk management, and institutional requirements, or expect some extra benefits from trading the derivatives such as portfolio insurance, gambling entertainment, and hedging benefits [see, for example, 67]. In other words, they exploit the liquidity of the derivative market and provide demand pressure to the market. Instead of modeling explicitly their various trading incentives, we assume that the consumers' aggregate demand of derivatives at time t, $D_t := (D_t^1, \cdots, D_t^n)$, is given exogenously. Note that such a demand D_t is observable directly from the market data [see, for example, 15].

The dealers provide liquidity to the derivatives market while bearing various inventory risks. They satisfy their costumers' demand of derivatives through competing with each other. We assume that, at time t, there exist a finite number of dealers indexed by I_t, who have *constant absolute risk aversion* (CARA). That is, they have exponential utility functions

$$U_i(x) = -e^{-\gamma_i x}, \quad x \in \mathbb{R}, \tag{2.5}$$

with $\gamma_i > 0$ being the risk aversion parameter of dealer $i \in I_t$. We shall also use the *risk tolerance parameter* $\tau_i = \frac{1}{\gamma_i}$ because it provides more intuition in interpreting our pricing formula. The objective of the dealers is to maximize their expected utility of terminal wealth.

Now suppose that, with initial endowment of x_i at time t, each dealer i takes static positions $q_i := (q_i^1, \cdots, q_i^n)$ in the derivatives $G = (G^1, \cdots, G^n)$. Since the market is competitive, the price vector $p_t = (p_t^1, \cdots, p_t^n)$ of the derivatives is taken as given, and the derivative positions thus cost $(q_i \cdot p_t)$ with "$\cdot$" being the usual inner product on $\mathbb{R}^n$. The dealer i then invests the remaining capital of $(x_i - q_i \cdot p)$ into the underlyings using an optimal trading strategy π_i, whilst collecting a random derivative payoff $q_i \cdot G$ at maturity T. At time t, each dealer i wants to maximize the expected utility of terminal wealth over both the static derivative positions q_i and the dynamic trading strategy π_i in the underlyings, and therefore solves the following optimal investment problem:

$$\underset{\pi_i \in \mathcal{A}, q_i \in \mathbb{R}^n}{\text{maximize}} \; E\left[U_i \left(X_T^{\pi_i, x_i - q_i \cdot p_t} + q_i \cdot G \right) \middle| \mathcal{F}_t \right], \tag{2.6}$$

where $\mathcal{A}$ is the set of admissible trading strategies and $i \in I_t$. Since we are interested in the equilibrium prices, we assume the existence of the optimal investment strategies and use them to define the competition-based derivative prices. The technical assumptions to ensure such existence will be specified later on when solving the equilibrium.

2.2.3 The Competitive Equilibrium

The derivative prices are determined through a competitive equilibrium among the dealers and their costumers. Precisely we introduce the following definition.

Definition 2.2.1. *The competitive price of the derivatives G is defined as vector $p(G, D)$ such that each dealer i admits an optimal position q_i^* in the derivatives and the derivative market clears. That is, the derivative price vector p satisfies the partial market clearing condition*

$$\sum_{i \in I_t} q_i^*(p) + D_t = 0, \tag{2.7}$$

where q_i^* is the solution to the optimal investment problem (2.6) for dealer $i \in I_t$.

In the above definition, the derivative prices are set based on the partial equilibrium arguments. No market clearing condition is imposed on the underlyings whose prices are given exogenously. It is a compromise between the no-arbitrage arguments in classical derivative pricing and the general equilibrium theory in asset pricing. Although it relies on the classical pricing principle of equilibrium from economic theory, this pricing notion is relatively new in derivatives pricing, in that it introduces competition and the effect demand and inventory into the pricing mechanism.

2.3 The Competition-Based Derivative Prices

We devote this section to construct the competition-based prices $p(G, D)$ for the derivatives $G = (G^1, \cdots, G^n)$ with payoffs $G^j = g^j(Y_T, Z_T)$, $(j = 1, \cdots, n)$, where

$$Z_s = z + \int_t^s \zeta(u, Y_u)du, \quad t \leq s \leq T. \tag{2.8}$$

For convenience we denote function $g := (g^1, \cdots, g^n)$. This payoff specification allows us to price a variety of volatility derivatives that are actively traded in the market, such as *variance swaps*, *volatility swaps*, *variance calls and puts*, and *volatility calls and puts.*

We start from investigating the optimal investment problem of the dealers. A useful observation is that the optimization problem (2.6) can be written as

$$\max_{q_i \in \mathbb{R}^n} u(t, x_i - q_i \cdot p, y, z; q_i, \gamma_i), \tag{2.9}$$

where the value function u is given by [3]

$$u(t,x,y,z,;q,\gamma) := \sup_{\pi \in \mathcal{A}} E_t^{x,y,z} \left[-e^{-\gamma\left(X_T^{\pi,x} + q\cdot g(Y_T, Z_T)\right)} \right], \tag{2.10}$$

with $X^{\pi,x}$ being the wealth dynamics defined in (2.3) and $\mathcal{A}$ the set of admissible trading strategies.

For fixed allocation q, the valuation problem (2.10) evaluates the maximal utility of terminal wealth arising from the derivative payoff $q \cdot g(Y_T, Z_T)$ and the terminal fund that can be achieved by trading optimally in the underlyings with initial capital x. Such valuation problem and its variations have been studied extensively in mathematical finance, in particular, in the context of indifference pricing. Stochastic control is one of the most popular approaches, which studies the problem through the associated Hamilton-Jacobi-Bellman (HJB) equations. For our present problem (2.10), the associated HJB equation is given by

$$u_t + \max_{\pi} \left(\frac{1}{2}\sigma^2 \pi^2 u_{xx} + \pi \left(\rho\sigma a u_{xy} + \mu u_x \right) \right) + \mathcal{L}^y u + \zeta u_z = 0, \tag{2.11}$$

for $(t,x,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, with $u(T,x,y,z) = -e^{-\gamma(x + q\cdot g(y,z))}$, where the differential operator $\mathcal{L}^y := \frac{1}{2}a^2 \frac{\partial^2}{\partial y^2} + b\frac{\partial}{\partial y}$.

It follows from a standard argument of viscosity solution that the solutions of the HJB equation (2.11) must coincide with the respective value function defined by (2.10). Moreover, following similar arguments used in Theorems 4.1 and 4.2 of [46], we find that the value function u defined in (2.10) is the unique viscosity solution of (2.11) in the class of functions that are concave, increasing in x, and uniformly bounded in y for any fixed (t,x). We now proceed to solve the HJB equation (2.11) following the similar approach used in [111].

[3] Hereafter, we adopt the short notation $E_t^{x,y,z}[\cdot] := E_{\mathbb{P}}[\cdot | X_t = x, Y_t = y, Z_t = z]$ for the conditional expectation under the historical measure $\mathbb{P}$.

Proposition 2.3.1. *The value function u can be further represented by*

$$u(t,x,y,z;q,\gamma) = -e^{-\gamma x} f(t,y,z;q,\gamma)^{\frac{1}{\bar{\rho}^2}}, \tag{2.12}$$

where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ solves the linear equation

$$f_t + \frac{1}{2}a^2(t,y)f_{yy} + (b(t,y) - \rho\lambda a(t,y))\, f_y + \zeta f_z = \frac{1}{2}\bar{\rho}^2\lambda^2(t,y)f, \tag{2.13}$$

with $f(T,y,z;q,\gamma) = e^{-\gamma\bar{\rho}^2 q\cdot g(y,z)}$ for all $(y,z) \in \mathbb{R} \times \mathbb{R}$.

Proof. We first observe that the second term of the HJB equation (2.11) is quadratic in π, whose maximum is achieved at

$$\pi^* = -\frac{\mu u_x + \rho\sigma a u_{xy}}{\sigma^2 u_{xx}}. \tag{2.14}$$

Substituting the optimal control π^* into (2.11) yields

$$u_t - \frac{(\mu u_x + \rho\sigma a u_{xy})^2}{2\sigma^2 u_{xx}} + \mathcal{L}^y u + \zeta u_z = 0. \tag{2.15}$$

Using the scaling property of the utility function and the structure of the wealth dynamics (2.3), we postulate an Ansatz solution $u(t,x,y,z) = -e^{-\gamma x}F(t,y,z)$. Plugging it into (2.15) yields the following quasilinear equation

$$F_t + (b - \rho\lambda a)\, F_y + \frac{1}{2}a^2 F_{yy} - \frac{1}{2}\rho^2 a^2 \frac{F_y^2}{F} + \zeta F_z = \frac{1}{2}\lambda^2 F.$$

We further substitute $F(t,y,z) = [f(t,y,z)]^\kappa$ for some constant κ and derive equation

$$f_t + (b - \rho\lambda a)\, f_y + \frac{1}{2}a^2 f_{yy} + \left(\frac{(\kappa-1)a^2}{2} - \frac{\kappa\rho^2 a^2}{2}\right)\frac{f_y^2}{f} + \zeta f_z = \frac{1}{2\kappa}\lambda^2 f,$$

which clearly can be linearized by choosing $\kappa = \frac{1}{1-\rho^2} = \frac{1}{\bar{\rho}^2}$. This leads to the linear equation (2.13). Combining the transformation we have used, we conclude the representation (2.12). □

To prepare for the probabilistic representation of the value function u, we recall that under stochastic volatility model the density of an *equivalent local mar-*

tingale measure (ELMM) Q is given by

$$\frac{dQ}{dP} = \exp\left\{-\int_0^T \lambda(s,Y_s)dW_s^1 - \int_0^T \varphi_s dW_s^{\perp} - \frac{1}{2}\int_0^T \left(\lambda^2(s,Y_s)+\varphi_s^2\right)ds\right\}, \quad (2.16)$$

where P is the historical measure, λ is the Sharpe ratio process of the stock, and φ is an adapted process satisfying $\int_0^T \varphi_s^2 ds < +\infty$ a.s. [see 12]. We assume that $E[dQ/dP]=1$ so that Q is a probability measure equivalent to P on $\mathcal{F}_T$. A sufficient condition to ensure this is the Novikov condition

$$E\left[\exp\left\{\frac{1}{2}\int_0^T \lambda^2(s,Y_s)ds\right\}\right] < +\infty. \quad (2.17)$$

We denote by $\mathcal{M}$ the set of all ELMM. It is worth noting that the set $\mathcal{M}$ is one-to-one correspondence to the set Φ of integrands φ. We write Q_φ to emphasis the dependence of Q on φ whenever needed. Under measure $Q \in \mathcal{M}$, the dynamics of S and Y are given by

$$\begin{aligned} dS_t =& \sigma(t,Y_t)S_t dW_t^{1,Q}, \\ dY_t =& \left(b(t,Y_t) - a(t,Y_t)(\rho\lambda(t,Y_t)+\bar{\rho}\varphi_t)\right)dt \\ &+ a(t,Y_t)\left(\rho dW_t^{1,Q} + \bar{\rho} dW_t^{\perp,Q}\right), \end{aligned} \quad (2.18)$$

where $(W^{1,Q}, W^{\perp,Q})$ is a two-dimensional standard Brownian motion defined by

$$W_t^{1,Q} := W_t^1 + \int_0^t \lambda(s,Y_s)ds, \quad (2.19)$$

$$W_t^{\perp,Q} := W_t^{\perp} + \int_0^t \varphi_s ds. \quad (2.20)$$

Among all ELMM, the *minimal entropy martingale measure* (MEMM) is closely related to the exponential utility maximization [see 40]. We recall that the *relative entropy* $H(Q|P)$ of any probability measure Q with respect to P is defined as

$$H(Q|P) := \begin{cases} E\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\big|\mathcal{F}_t\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P. It is worth noting that the relative entropy $H(Q|P)$ measures the distance between the two probability distributions Q and P, which is always nonnegative [see 75]. The entropy of an ELMM $Q \in \mathcal{M}$ is

$$H(Q_\varphi|P) = E_{Q_\varphi}\left[\frac{1}{2}\int_0^T \left(\lambda^2(s,Y_s) + \varphi_s^2\right) ds\right]. \tag{2.21}$$

Obviously, there exists an ELMM with finite entropy. Thus there is a unique martingale measure $Q_e \in \mathcal{M}$ minimizing the relative entropy $H(Q|P)$ over all $Q \in \mathcal{M}$, nemely

$$Q_e = \arg\min_{Q\in\mathcal{M}} H(Q|P). \tag{2.22}$$

See Theorem 2.2 and 2.3 of [54] and Proposition 3.2 and the proof of Theorem 4.3 of [64].

Another important martingale measure arising from our valuation problem is the so called *minimal martingale measure* (MMM) Q_0, whose density is given by

$$\frac{dQ_0}{dP} = \exp\left\{-\int_0^T \lambda(s,Y_s)dW_s^1 - \frac{1}{2}\int_0^T \lambda^2(s,Y_s)ds\right\}, \tag{2.23}$$

which corresponds to take $\varphi = 0$ in (2.16). This measure was originally introduced by [52]. Under measure Q_0 the discounted traded asset is a martingale while the law of the orthogonal martingale measure remains unchanged. For models with continuous price process, it has been shown that Q_0 is the martingale measure minimizing the reverse entropy $H(\mathbb{P}|Q)$ over all ELMM $Q \in \mathcal{M}$, i.e.

$$Q_0 = \arg\min_{Q\in\mathcal{M}} H(P|Q) := \arg\min_{Q\in\mathcal{M}} E_P\left(-\log\frac{dQ}{dP}\right). \tag{2.24}$$

We refer the reader to [125, 126] for more discussions. We are now ready to prove the following representation of the value function u.

Theorem 2.3.2. *The value function u is given by*

$$u(t,x,y,z;q;\gamma) = -e^{-\gamma(x+w(t,y,z;q;\gamma))}, \tag{2.25}$$

where

$$w(t,y,z;q;\gamma) =$$
$$-\frac{1}{\gamma\bar{\rho}^2}\log\left(E_{Q_e}^{t,y,z}\left[e^{-\gamma\bar{\rho}^2 q\cdot g(Y_T,Z_T)}\right]E_{Q_0}^{t,y,z}\left[\exp\left\{-\frac{\bar{\rho}^2}{2}\int_t^T\lambda^2(s,Y_s)ds\right\}\right]\right), \quad (2.26)$$

with Q_e *being the minimal entropy martingale measure defined by*

$$\left.\frac{dQ_e}{dP}\right|_{\mathcal{F}_t} = \frac{\exp\left\{-\int_t^T\lambda(s,Y_s)dW_s^1 - \frac{1}{2}(1+\bar{\rho}^2)\int_t^T\lambda^2(s,Y_s)ds\right\}}{E^{t,y}\left[\exp\left\{-\int_t^T\lambda(s,Y_s)dW_s^1 - \frac{1}{2}(1+\bar{\rho}^2)\int_t^T\lambda^2(s,Y_s)ds\right\}\right]}. \quad (2.27)$$

Proof. By the Girsanov's theorem, we see that, under the MMM Q_0 defined in (2.23), the process $\tilde{W}_t^1 = W_t^1 + \int_0^t\lambda(s,Y_s)ds$ is a standard Brownian motion independent with $W_t^{\perp}$. It follows that the dynamics of S, Y and Z under Q_0 are given by

$$dS_t = \sigma(t,Y_t)S_t d\tilde{W}_t^1, \quad (2.28)$$
$$dY_t = (b(t,Y_t) - \rho\lambda a(t,Y_t))dt + a(t,Y_t)\left(\rho d\tilde{W}_t^1 + \bar{\rho}dW_t^{\perp}\right), \quad (2.29)$$
$$dZ_t = \zeta(t,Y_t)dt, \quad (2.30)$$

for $t \geq 0$. Using the Feynman-Kăc formula, we find that the solution $f(t,y,z)$ to the equation (2.13) admits the probabilistic representation

$$f(t,y,z) = E_{Q_0}^{t,y,z}\left[e^{-\gamma\bar{\rho}^2 q\cdot g(Y_T,Z_T)}\exp\left\{-\frac{\bar{\rho}^2}{2}\int_t^T\lambda^2(s,Y_s)ds\right\}\right], \quad (2.31)$$

for $(t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}$. We next define a measure Q_e through density

$$\left.\frac{dQ_e}{dQ_0}\right|_{\mathcal{F}_t} = \frac{\exp\left\{-\frac{\bar{\rho}^2}{2}\int_t^T\lambda^2(s,Y_s)ds\right\}}{E_{Q_0}^{t,y}\left[\exp\left\{-\frac{\bar{\rho}^2}{2}\int_t^T\lambda^2(s,Y_s)ds\right\}\right]}, \quad (2.32)$$

which is sometimes called Esscher change of measure. Combining (2.31) and (2.32), we find that

$$f(t,y,z;q;\gamma) = E_{Q_e}^{t,y,z}\left[e^{-\gamma\bar{\rho}^2 q\cdot g(Y_T,Z_T)}\right]E_{Q_0}^{t,y}\left[\exp\left\{-\frac{\bar{\rho}^2}{2}\int_t^T\lambda^2(s,Y_s)ds\right\}\right].$$

Now denote $w(t,y,z;q,\gamma) := -\frac{1}{\gamma\bar{\rho}^2}\log f(t,y,z;q,\gamma)$, then w can be written as (2.26) and $f(t,y,z;q,\gamma) = e^{-\gamma\bar{\rho}^2 w(t,y,z;q,\gamma)}$. Therefore, we conclude the representation (2.25) by plugging f into (2.12). The fact that Q_e is the minimal entropy martingale measure has been well established; see [54, 6, 64, 108]. We finally combine (2.23) and (2.32) to deduce the density (2.27). $\square$

It is useful to observe that the function $w(t,y,z;q,\gamma)$ is strictly concave in q for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. This follows from the convexity of the function

$$F(\alpha) = \log E_{\mathbb{P}}\left[e^{-\alpha\cdot G}\right].$$

To see that, we calculate

$$\begin{aligned}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}F(\alpha) &= \frac{E_{\mathbb{P}}\left[e^{-\alpha\cdot G}G_iG_j\right]}{E_{\mathbb{P}}\left[e^{-\alpha\cdot G}\right]} - \frac{E_{\mathbb{P}}\left[e^{-\alpha\cdot G}G_i\right]}{E_{\mathbb{P}}\left[e^{-\alpha\cdot G}\right]}\frac{E_{\mathbb{P}}\left[e^{-\alpha\cdot G}G_j\right]}{E_{\mathbb{P}}\left[e^{-\alpha\cdot G}\right]} \\ &= E_{\mathbb{Q}}\left[G_iG_j\right] - E_{\mathbb{Q}}\left[G_i\right]E_{\mathbb{Q}}\left[G_j\right] \\ &= \mathrm{Cov}_{\mathbb{Q}}(G_i, G_j),\end{aligned}$$

where the measure $\mathbb{Q}$ is given by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{-\alpha\cdot G}}{E\left[e^{-\alpha\cdot G}\right]}.$$

Therefore, the convexity of the function F follows from the fact that the covariance matrix of G under measure $\mathbb{Q}$ is positive semi-definite. Moreover, since the derivatives G are non-redundant, the covariance matrix is indeed positive definite, which implies that the function w is strictly concave in q.

Now we are ready to prove the following results.

Theorem 2.3.3. *The dealers' optimal investment problems (2.9) admit a unique solution q_i. Moreover, the optimal derivative holdings q_i are characterized by the first-order condition*

$$E^{t,y,z}_{Q_e}\left[e^{-\gamma_i\bar{\rho}^2 q_i\cdot g(Y_T,Z_T)}\big(g(Y_T,Z_T) - p_t\big)\right] = 0, \tag{2.33}$$

with Q_e being the minimal entropy martingale measure defined in (2.27).

Proof. We first observe from (2.9) and (2.25) that

$$\max_{q_i \in \mathbb{R}^n} u\left(t, x_i - q_i \cdot p, y, z; q_i, \gamma_i\right) = -e^{-\gamma_i x_i} \exp\left\{-\gamma_i \max_{q_i \in \mathbb{R}^n} \left(w(t, y, z; q_i, \gamma_i) - q_i \cdot p\right)\right\},$$

where w is given in (2.26). Thus, the existence and uniqueness of q_i follow directly from the concavity of w. Moreover, the first-order condition of u is given by

$$0 = \nabla_q u(t, x_i - q_i \cdot p, y, z; q_i, \gamma_i) = \gamma u \left[p - \nabla_q w(t, y, z; q_i, \gamma_i)\right],$$

which yields that

$$p = \nabla w(t, y, z; q_i, \gamma_i) = \frac{E_{Q_e}^{t,y,z}\left[e^{-\gamma_i \bar{\rho}^2 q_i \cdot g(Y_T, Z_T)} g(Y_T, Z_T)\right]}{E_{Q_e}^{t,y,z}\left[e^{-\gamma_i \bar{\rho}^2 q_i \cdot g(Y_T, Z_T)}\right]}, \tag{2.34}$$

and completes the proof. □

In the next theorem, we summarize the competition-based pricing formula and the associated pricing measure.

Theorem 2.3.4. *Let τ denote the aggregate risk tolerance parameter of the market makers in the economy, namely, $\tau := \sum_{i \in I_t} \tau_i = \sum_{i \in I_t} \frac{1}{\gamma_i}$, then the competition-based price $p(G, D)$ is given by*

$$p_t = E_{Q_*}\left[g(Y_T, Z_T) | Y_t = y, Z_t = z\right], \tag{2.35}$$

where the pricing measure Q_* with density given by

$$\left.\frac{dQ_*}{dQ_e}\right|_{\mathcal{F}_t} = \frac{e^{\frac{\bar{\rho}^2}{\tau} D \cdot g(Y_T, Z_T)}}{E_{Q_e}^{t,y,z}\left[e^{\frac{\bar{\rho}^2}{\tau} D \cdot g(Y_T, Z_T)}\right]}, \tag{2.36}$$

which is an Esscher transform of the minimal entropy martingale measure Q_e.

Proof. We first take the price vector p_t as given and denote by α_p the unique solution to the equation

$$E_{Q_e}^{t,y,z}\left[e^{-\bar{\rho}^2 \alpha \cdot g(Y_T, Z_T)}(g(Y_T, Z_T) - p_t)\right] = 0,$$

where Q_e is given by (2.27). Thus the first-order condition (2.33) shows that the optimal derivative positions q_i of the intermediary i satisfy $\gamma_i q_i = \alpha_p$, i.e.

$$q_i = \frac{\alpha_p}{\gamma_i} = \alpha_p \tau_i.$$

Combining the market clearing condition (2.7) yields that

$$\alpha_p = -\frac{D}{\sum_{i \in I_t} \tau_i} = -\frac{D}{\tau}.$$

To this end, the first-order condition (2.33) again shows that the equilibrium price p_t is given by

$$p_t = \frac{E_{Q_e}^{t,y,z}\left[e^{\frac{\bar{\rho}^2}{\tau} D \cdot g(Y_T, Z_T)} g(Y_T, Z_T)\right]}{E_{Q_e}^{t,y,z}\left[e^{\frac{\bar{\rho}^2}{\tau} D \cdot g(Y_T, Z_T)}\right]}. \tag{2.37}$$

Therefore, we conclude by recalling the definition of Q_*. □

Remark 2.3.5. *The pricing measure Q_* can be as well characterized as follows. Define the probability measure $P_* \sim P$ through the Esscher transform*

$$\left.\frac{dP_*}{dP}\right|_{\mathcal{F}_t} = \frac{e^{\frac{1}{\tau} D \cdot g(Y_T, Z_T)}}{E_t^{y,z}\left[e^{\frac{1}{\tau} D \cdot g(Y_T, Z_T)}\right]}, \tag{2.38}$$

where D is the aggregate demand of the derivatives G. Then the pricing measure Q_* coincides with the minimal entropy martingale measure with respect to the new prior P_*, namely,

$$Q_* = \underset{Q \in \mathcal{M}}{\arg\min}\, H(Q|P_*) := \underset{Q \in \mathcal{M}}{\arg\min}\, E_Q\left[\log \frac{dQ}{dP_*}\right]. \tag{2.39}$$

We refer to Su (2006) for further discussion of this characterization under a semi-martingale setting.

From the above result follows immediately that for $D_t = 0$ the pricing measure Q_* becomes identical to the minimal entropy martingale measure Q_e. In other words, the competition-based price reduces to the marginal utility price. This result can be stated as follows.

Corollary 2.3.6. *When the aggregate demand $D = 0$, the competition-based price reduces to the marginal utility price, namely, $p_t(G, 0) = E_{Q_e}\left[g(Y_T, Z_T)|Y_t = y, Z_t = z\right]$.*

The Price Effects of Demand and Inventory. In the sequel, we investigate the effects of demand and inventory on the derivative prices. We first introduce the following pricing kernel

$$\xi(D; \tau, G) := \frac{dQ_e}{dP} \cdot \frac{dQ_*}{dQ_e}, \tag{2.40}$$

where $\frac{dQ_e}{dP}$ and $\frac{dQ_*}{dQ_e}$ are given by (2.27) and (2.36), respectively.

From Theorem 2.3.4 follows that, in terms of the pricing kernel $\xi(D; G)$, the competitive price p^j of the derivative $G^j = g^j(Y_T, Z_T)$ is given by

$$p^j = E_t^{y,z}\left[\xi(D, G)G^j\right] = E_t^{y,z}\left[\xi(D, G)g^j(Y_T, Z_T)\right].$$

We next prove the following sensitivity result.

Theorem 2.3.7. *The partial derivative of p_j, $(j = 1, \cdots, n)$, with respect to the demand D_k of the derivative G^k, $(k = 1, \cdots, n)$, is given by*

$$\frac{\partial p^j}{\partial D^k} = \frac{\bar{\rho}^2}{\tau} Cov_{Q_*}^{t,y,z}\left(G^j, G^k\right) = \frac{1}{\tau} Cov_{Q_*}^{t,y,z}\left(\tilde{G^j}, \tilde{G^k}\right), \tag{2.41}$$

where $\tilde{G^j} = \bar{\rho}G^j$ is the unhedgeable part of the claim G^j, $(j = 1, \cdots, n)$.

Proof. We recall that under the pricing kernel $\xi(D)$, the competitive price

$$p_j = E\left[\xi(D)G^j\middle|\mathcal{F}_t\right] = E_{Q_e}^{t,y,z}\left[\frac{dQ_*}{dQ_e}G^j\right].$$

We then differentiate the above equation with respect to D_k to deduce

$$
\begin{aligned}
\frac{\partial p_j}{\partial D_k} &= E_{Q_e}^{t,y,z}\left[\frac{\partial}{\partial D_k}\left(\frac{dQ_*}{dQ_e}\right)G^j\right] \\
&= \frac{\bar{\rho}^2}{\tau}E_{Q_e}^{t,y,z}\left[\frac{dQ_*}{dQ_e}\left(G^k - E_{Q_e}^{t,y,z}\left[\frac{dQ_*}{dQ_e}G^k\right]\right)G^j\right] \\
&= \frac{\bar{\rho}^2}{\tau}E_{Q_*}^{t,y,z}\left[G^jG^k - G^jE_{Q_*}^{t,y,z}\left[G^k\right]\right] \\
&= \frac{\bar{\rho}^2}{\tau}\left(E_{Q_*}^{t,y,z}\left[G^jG^k\right] - E_{Q_*}^{t,y,z}\left[G^j\right]E_{Q_*}^{t,y,z}\left[G^k\right]\right) \\
&= \frac{\bar{\rho}^2}{\tau}\mathrm{Cov}_{Q_*}^{t,y,z}\left(G^j, G^k\right) \\
&= \frac{1}{\tau}\mathrm{Cov}_{Q_*}^{t,y,z}\left(\tilde{G}^j, \tilde{G}^k\right),
\end{aligned}
$$

as desired. $\square$

The above theorem shows that the price p^j of a derivative G^j is increasing with the demand D^k of another derivative G^k in a rate proportional to the covariance, under the competitive pricing measure, between the unhedgeable parts of the two associated derivative payoffs. This result helps to explain the *options-pricing puzzles* because it is evident that the demands of index options and deep out-of-the-money (OTM) puts are very high comparing to other individual equity options; see, among others, [15, 21, 22, 56].

2.4 Properties of The Competition-Based Pricing

In this section, we investigate the properties of the competition-based prices. First of all, we observe that the competition-based pricing kernel (measure) depends on the market parameters $\lambda, \rho, G, D, \tau$. It is, however, independent of the individual dealer's initial capital endowment and does not explicitly depend on the individual's preference.

No Arbitrage. It is obviously that the competition-based price consists with the no-arbitrage principle because

$$
\inf_{Q\in\mathcal{M}} E_Q[B] \le E_{Q_*}[B] \le \sup_{Q\in\mathcal{M}} E_Q[B],
$$

for any bounded claim $B = \alpha \cdot g(Y_T, Z_T)$ since $Q_* \in \mathcal{M}$.

Scaling Invariance. From the definition (2.40), we can easily verify the following scaling property for the pricing kernel

$$\xi(kD, k\tau; G) = \xi(D, \tau; G).$$

Monotonicity. For any claim $B = \alpha \cdot g(Y_T, Z_T)$, the competitive price is increasing in B:

$$p(B_1; D) \leq p(B_2; D) \quad \text{if} \quad B_1 \leq B_2.$$

Price Effects of Risk Tolerance τ. Following the same line of arguments in the proof of Theorem 2.3.7, one can easily shows that

$$\frac{\partial}{\partial \tau} p(B; \tau) = -\frac{\bar{\rho}^2}{\tau^2} Cov_{Q_*}^{t,y,z}(B, D \cdot G).$$

Thus, we have the following monotonicity of the competitive price for any claim $B \geq 0$.

- *Decreasing in τ for positive demand*: $p(B; \tau_1) \geq p(B; \tau_2)$ if $\tau_1 \leq \tau_2$ and $D \cdot G \geq 0$.
- *Increasing in τ for negative demand*: $p(B; \tau_1) \leq p(B; \tau_2)$ if $\tau_1 \leq \tau_2$ and $D \cdot G \geq 0$.

Robustness and Regularity. From the competitive pricing measure (2.36), we see that, $\frac{dQ_*}{dQ_e} \to 1$ a.s. as $\tau \to \infty$. This leads to

- *Regularity with respect to τ:*

$$\lim_{\tau \to \infty} p(G; \tau) = E_{Q_e}\left[g(S_T, Y_T) \middle| Y_t = y, Z_t = z\right].$$

In other words, we recover the marginal utility price when risk tolerance $\tau \to \infty$. Similarly, as $|\rho| \to 1$, $\bar{\rho}^2 \to 0$ and further $\frac{dQ_*}{dQ_0} \to 1$ a.s., which follows

- *Regularity with respect to ρ:*

$$\lim_{|\rho|\to 1} p(G;\rho) = E_{Q_0}\left[g(S_T, Y_T) \middle| Y_t = y, Z_t = z \right].$$

We further consider an asymptotic complete market, where the asset Y is perfectly correlated with the security S and $\lambda = \frac{b}{a}$. That is, the security S becomes the perfect proxy of the asset Y, which can be viewed as if it is tradable. In this case, the measure Q_0 becomes the unique martingale measure, which leads to

- *Robustness:* If the tradable asset S becomes the perfect proxy of the nontraded asset Y, namely, $|\rho| = 1$ and the *Sharpe ratios* $\frac{b}{a} \equiv \frac{\mu}{\sigma}$, then the competitive price p reduces to the Black-Scholes price.

Relation to Indifference Pricing. The notion of indifference pricing was introduced to price European options by [72] and further extended by [35]. The valuation methodology is based on the comparison between two optimal investment problems with and without involving the derivative. Under the current market setting, the first relevant optimization problem is the classical Merton model. The value function is defined as

$$M(t,x,y) := \sup_{\mathcal{A}} E_t^{x,y}\left[U(X_T)\right], \tag{2.42}$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, where the wealth process (X_s) satisfies (2.3) and $\mathcal{A}$ is the set of admissible policies. The second optimal investment problem involves the terminal liability defined by the claim $C = q \cdot g(Y_T, Z_T)$. The corresponding value function is defined by

$$V(t,x,y,z) := \sup_{\mathcal{A}} E_t^{x,y,z}\left[U(X_T - g(Y_T, Z_T))\right], \tag{2.43}$$

for $(t,x,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, where the wealth process (X_s) satisfies (2.3) and $\mathcal{A}$ is the set of admissible policies.

We next follow [111] to define the seller's indifference price using the above value functions.

Definition 2.4.1. *The seller's indifference price for the derivative $C = q \cdot g(Y_T, Z_T)$*

is defined as the amount $h(t,x,y,z;q)$ *such that*

$$M(t,x,y) = V(t, x + h(t,x,y,z;q), y, z), \tag{2.44}$$

for $(t,x,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$*, where the value functions* M *and* V *are defined in (2.42) and (2.43), respectively.*

Under exponential utility (2.5), we derive the indifference price formula (2.45), which is an analogue of Theorem 3 in [111].

Proposition 2.4.2. *The seller's indifference price is given by*

$$h(t,y,z;q) = \frac{1}{\gamma\bar{\rho}^2} \log E_{Q_e}^{t,y,z}\left[e^{\gamma\bar{\rho}^2 q\cdot g(Y_T,Z_T)}\right], \tag{2.45}$$

for $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+$*, where the pricing measure* Q_e *is the minimal entropy martingale measure defined in (2.27).*

Now it is straightforward to verify the following result.

Proposition 2.4.3. *Let* $\nu(q;G) = h(t,y,z;q)$ *denote the indifference price for the claim* $C = q \cdot g(Y_T, Z_T)$*, then the competitive prices of the derivative* $g^j(Y_T,Z_T)$ *is given by*

$$p_j(G,D) = \frac{\partial}{\partial q_j}\nu(D;G). \tag{2.46}$$

The above result shows that the competitive prices of the derivatives $G = g(Y_T, Z_T)$ equal to the seller's marginal indifference price.

Relation to Marginal Utility Pricing. The marginal indifference price was first introduced by [32] based on the idea of "zero marginal rate of substitution".[4] Considering a representative agent economy and a given contingent claim, the fair price $\hat{p}$ is defined such that the agent can not increase his/her expected utility by diverting an infinitesimal amount of capital into the claim. Formally, we follow [32] to define the marginal utility price.

[4]There are many terminologies arising from this pricing concept such as "marginal (indifference) price", "(marginal) utility-based price", "Davis' fair price", "neutral price" as in [79, 80], and "shadow price" as in [51].

Definition 2.4.4. *Consider a single European claim B and a representative agent who wants to maximize the expected utility of terminal wealth:*

$$u(x, \delta; B) = \sup_{\vartheta \in \Theta} E\left[U(X_T^{\vartheta,x} + \delta B)\right],$$

where $U(x)$ is the agent's utility function satisfying the Inada condition. Then the marginal price $\hat{p}(B)$ of the claim B satisfies

$$\frac{\partial}{\partial \delta} u(x - \delta \hat{p}, \delta; B)\Big|_{\delta=0} = 0.$$

Under exponential utility (2.5), it has been shown that the fair price is given by

$$\hat{p} = E_{Q_e}[B],$$

for a bounded claim B. This coincides with the competition-based price when we set the demand pressure equal zero and the aggregate risk tolerance parameter equal $1/\gamma$ (cf Corollary 2.3.6). Alternatively, the marginal utility price can be viewed as the limit of the competitive price as risk tolerance $\tau \to \infty$. Therefore the concept of competition-based price generalizes the framework of marginal utility price. In our framework, the marginal price can be interpreted as a special case of the competition-based price. That corresponds to the case of single agent, single derivative, and zero demand pressure.

2.5 Applications to Volatility Derivatives

In this section, we illustrate an application of our model to price volatility derivatives. We consider the Heston stochastic model [see 69], which corresponds to choose $\mu(t, Y_t) = \mu(t)$ and $\sigma(t, Y_t) = \sqrt{Y_t}$ in equation (2.1), and $b(t, Y_t) = -\eta(Y_t - \bar{Y})$ and $a(t, Y_t) = \delta\sqrt{Y_t}$ in equation (2.2). Then the dynamics of the discounted stock price and the stochastic factor become

$$dS_t = (\mu(t) - r)S_t dt + \sqrt{Y_t} S_t dW_t^1, \quad t \geq 0, \tag{2.47}$$

$$dY_t = -\eta(Y_t - \bar{Y})dt + \delta\sqrt{Y_t} dW_t, \quad t \geq 0, \tag{2.48}$$

where η is the speed of reversion of Y_t to its long-term mean $\bar{Y}$, and $W_t = \rho W_t^1 + \bar{\rho} W_t^{\perp}$. The process followed by the instantaneous variance Y_t may be recognized as a version of the so-called CIR process introduced by [29]. The CIR process and its variants have been widely used to model the interest rate term structure, in particular, the *affine* term structure model.

We focus on the following four types of payoffs of particular financial interest: (i) variance or volatility swap with the discounted payoff given by

$$e^{-r(T-t)} \int_t^T Y_s ds \quad \text{or} \quad e^{-r(T-t)} \left(\int_t^T Y_s ds \right)^{\frac{1}{2}}, \tag{2.49}$$

(ii) European calls or puts on realized variance with discounted payoff

$$e^{-r(T-t)} \left(\int_t^T Y_s ds - K \right)^+ \quad \text{or} \quad e^{-r(T-t)} \left(K - \int_t^T Y_s ds \right)^+, \tag{2.50}$$

(iii) volatility calls or puts with discounted payoff

$$e^{-r(T-t)} \left(\sqrt{\int_t^T Y_s ds} - K \right)^+ \quad \text{or} \quad e^{-r(T-t)} \left(K - \sqrt{\int_t^T Y_s ds} \right)^+, \tag{2.51}$$

Unless otherwise specified, we use the well-known BCC parameters [see 7] for the Heston model, which are specified in Table 2.1. There we also set the default parameters for aggregate risk tolerance τ and time to maturity T.

Table 2.1: Heston model parameters.

r	μ	Y_0	$\bar{Y}$	η	δ	ρ	τ	T
4.0%	12.7%	0.04	0.04	1.15	39%	-28%	10	1.0

From (2.28) and (2.29) it follows that the dynamics of S and Y under the minimal martingale measure Q_0 are given by

$$d\tilde{S}_t = \sqrt{\tilde{Y}_t}\tilde{S}_t d\tilde{W}_t^1, \quad t \geq 0, \tag{2.52}$$

$$d\tilde{Y}_t = \big((\eta\bar{Y} - \rho\mu\delta) - \eta\tilde{Y}_t\big)dt + \delta\sqrt{\tilde{Y}_t}d\tilde{W}_t, \quad t \geq 0, \tag{2.53}$$

$$d\tilde{Z}_t = \tilde{Y}_t dt, \quad t \geq 0, \tag{2.54}$$

where $\tilde{W}_t = \rho \tilde{W}_t^1 + \bar{\rho} \tilde{W}_t^\perp$ with $(\tilde{W}^1, \tilde{W}^\perp)$ being a standard 2-dimentional Brownian motions under Q_0. Applying Theorem 2.3.4, we see that the derivative prices can be written as

$$p = E_{Q_0}\left[\tilde{\xi}(D) g(\tilde{Y}_T, \tilde{Z}_T)\right], \tag{2.55}$$

where the density $\tilde{\xi}(D)$ is given by

$$\tilde{\xi}(D) := \frac{dQ_*}{dQ_0} = \frac{\exp\left\{\frac{\bar{\rho}^2}{\tau} D \cdot g(\tilde{Y}_T, \tilde{Z}_T) - \frac{(\mu-r)^2\bar{\rho}^2}{2}\int_0^T \frac{1}{\tilde{Y}_s} ds\right\}}{E_{Q_0}\left[\exp\left\{\frac{\bar{\rho}^2}{\tau} D \cdot g(\tilde{Y}_T, \tilde{Z}_T) - \frac{(\mu-r)^2\bar{\rho}^2}{2}\int_0^T \frac{1}{\tilde{Y}_s} ds\right\}\right]}. \tag{2.56}$$

2.5.1 Numerical Treatments

In the sequel, we develop Monte Carlo simulation schemes to compute the competition-based price of volatility derivatives. The essential step in the simulation is to generate the paths of the CIR process given by

$$dY_t = (A(t) - B(t)Y_t)dt + C\sqrt{Y_t}dW_t, \quad t \geq 0, \tag{2.57}$$

with $Y_0 \geq 0$. It is well known that the above SDE (2.57) has a unique nonnegative solution provided that $A \geq 0$ and $C \geq 0$ [see 117]. If, in addition, we impose condition $A > C^2$, then the process is always positive [see 87]. Note also that the process is mean-reverting when $B > 0$, which is a desirable property for modeling stochastic volatility. The challenge for our simulation is to produce positive sample paths.

Milstein Discretization. It is straightforward to discretize the paths using a Milstein scheme. Specifically, by Ito-Taylor expansion, we have

$$Y_{t+\Delta t} - Y_t = (A - BY_t)\Delta t + C\sqrt{Y_t}\Delta W_t + \frac{C^2}{4}(\Delta W_t^2 - \Delta t).$$

It follows that

$$Y_{t+\Delta t} = \left(\sqrt{Y_t} + \frac{C}{2} w\sqrt{\Delta t}\right)^2 + \left(A - \frac{C^2}{4} - BY_t\right)\Delta t, \tag{2.58}$$

where $w \sim N(0,1)$. The problem of the Milstein scheme is that it does not guarantee a positive path, though it reduces significantly the negativity problem if comparing to the Euler scheme. In the sequel, we follow Alfonsi (2005) to develop schemes that guarantee the positivity of the CIR process.

The First Implicit Schemes. We first consider the following discretization

$$Y_{t+\Delta t} - Y_t = (A - BY_{t+\Delta t})\Delta t + C\sqrt{Y_{t+\Delta t}}\Delta W_t - \frac{C^2}{2}\Delta t,$$

which yields the quadratic equation

$$(1 + B\Delta t)Y_{t+\Delta t} - Cw\sqrt{\Delta t} \cdot \sqrt{Y_{t+\Delta t}} - \left[Y_t + \left(A - \frac{C^2}{2}\right)\Delta t\right] = 0.$$

The only positive root defines the first implicit scheme

$$\sqrt{Y_{t+\Delta t}} = \frac{Cw\sqrt{\Delta t} + \sqrt{C^2w^2\Delta t + 4(1+B\Delta t)\left(Y_t + \left(A - \frac{C^2}{2}\right)\Delta t\right)}}{2(1+B\Delta t)}. \tag{2.59}$$

The Second Implicit Schemes. By the Ito formula, we calculate

$$d\sqrt{Y_t} = \left(\frac{4A - C^2}{8\sqrt{Y_t}} - \frac{B}{2}\sqrt{Y_t}\right)dt + \frac{C}{2}dW_t, \quad t \geq 0.$$

Applying the Euler scheme implicitly in the drift gives

$$\sqrt{Y_{t+\Delta t}} - \sqrt{Y_t} = \left(\frac{4A - C^2}{8\sqrt{Y_{t+\Delta t}}} - \frac{B}{2}\sqrt{Y_{t+\Delta t}}\right)\Delta t + \frac{C}{2}\Delta W_t,$$

which implies the following quadratic equation in $\sqrt{Y_{t+\Delta t}}$:

$$\left(1 + \frac{B}{2}\Delta t\right)Y_{t+\Delta t} - \left(\sqrt{Y_t} + \frac{C}{2}w\sqrt{\Delta t}\right)\sqrt{Y_{t+\Delta t}} - \frac{4A - C^2}{8\sqrt{Y_{t+\Delta t}}}\sqrt{\Delta t} = 0.$$

The only positive root is then given by

$$\sqrt{Y_{t+\Delta t}} = \frac{\sqrt{Y_t} + \frac{C}{2}w\sqrt{\Delta t} + \sqrt{\left(\sqrt{Y_t} + \frac{C}{2}w\sqrt{\Delta t}\right)^2 + (2 + B\Delta t)\left(A - \frac{C^2}{4}\right)\Delta t}}{2 + B\Delta t}, \tag{2.60}$$

which provides the second implicit scheme for the CIR process.

Explicit Schemes. Applying Taylor expansion to the above two implicit schemes, we obtain the following explicit schemes

$$Y_{t+\Delta t} = \left[\left(1 - \frac{B}{2}\Delta t\right)\sqrt{Y_t} + \frac{Cw\sqrt{\Delta t}}{2 - B\Delta t}\right]^2 + \left(A - \frac{C^2}{4}\right)\Delta t + k\left(\Delta W_t^2 - \Delta t\right),$$

for $0 \leq k \leq A - \frac{C^2}{4}$, where we note that the factor $1 - B\Delta t/2$ can be replaced by $\sqrt{1 - B\Delta t}$. It follows that

$$Y_{t+\Delta t} = \left[\left(1 - \frac{B}{2}\Delta t\right)\sqrt{Y_t} + \frac{Cw\sqrt{\Delta t}}{2 - B\Delta t}\right]^2 + \left(A - \frac{C^2}{4} - k + kw^2\right)\Delta t, \tag{2.61}$$

or, by using the square-root factor instead, that

$$Y_{t+\Delta t} = \frac{\left(2(1 - B\Delta t)\sqrt{Y_t} + Cw\sqrt{\Delta t}\right)^2}{4(1 - B\Delta t)} + \left(A - \frac{C^2}{4} - k + kw^2\right)\Delta t, \tag{2.62}$$

for $0 \leq k \leq A - \frac{C^2}{4}$.

2.5.2 Numerical Results

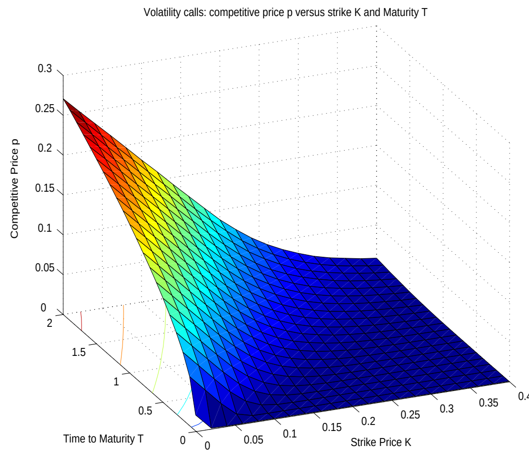
We first compare the above Monte Carlo schemes in compute the competitive prices of variance swaps. We use the BCC parameters given in Table 2.1, while the level of aggregate demand and inventory is set to be $1,000$ and the maturity ranges from 3 months to 2 years. The numerical results are shown in Table 2.2. The simulation is based on 10^6 (500,000 plus 500,000 antithetic) sample paths with a time step equal to 10^{-3}. As can be seen, these schemes, in particular, the explicit scheme $E(0)$, produce a fairly good estimation.

In what follows, we apply the explicit scheme $E(0)$ to compute the competitive prices of volatility derivatives and investigate their dependance on the market parameters. Figure 2.1 plots the competitive price of volatility calls (left) and variance puts (right) as a function of strike price K and maturity T. The level of aggregate demand and inventory is set to be -100, and the other parameters are listed in Table 2.1.

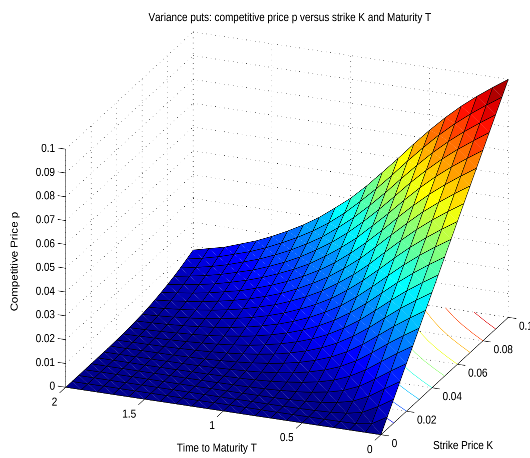
Table 2.2: Competitive prices using Monte Carlo simulations.

	Milstein Scheme			Explicit Scheme $E(0)$		
T	Price (s.e.)	90% C.I.	Time	Price (s.e.)	90% C.I.	Time
0.25	1.3404 (0.0012)	[1.3384, 1.3424]	121.846	1.3387 (0.0012)	[1.3367, 1.3406]	124.939
0.50	5.8537 (0.0396)	[5.7885, 5.9189]	255.187	5.9000 (0.0439)	[5.8277, 5.9723]	263.129
1.00	21.2048 (0.1112)	[21.0218, 21.3879]	532.535	21.1690 (0.1084)	[20.9905, 21.3475]	546.877
1.50	34.8614 (0.1847)	[34.5573, 35.1656]	804.467	34.5334 (0.1723)	[34.2497, 34.8170]	810.325
2.00	46.9262 (0.2428)	[46.5264, 47.3260]	1079.532	46.8576 (0.2391)	[46.4640, 47.2513]	1089.437
	First Implicit Scheme			Second Implicit Scheme		
T	Price (s.e.)	90% C.I.	Time	Price (s.e.)	90% C.I.	Time
0.25	1.3450 (0.0012)	[1.3431, 1.3470]	130.077	1.3392 (0.0012)	[1.3372, 1.3412]	124.969
0.50	5.8439 (0.0396)	[5.7788, 5.9091]	275.336	5.8448 (0.0401)	[5.7789, 5.9107]	271.250
1.00	20.9953 (0.1079)	[20.8176, 21.1730]	564.622	21.2971 (0.1190)	[21.1012, 21.4930]	559.324
1.50	34.8382 (0.1771)	[34.5466, 35.1298]	851.485	34.7191 (0.1836)	[34.4169, 35.0214]	845.125
2.00	46.9297 (0.2423)	[46.5308, 47.3287]	1139.298	47.0617 (0.2413)	[46.6644, 47.4590]	1133.851

This table compares the four Monte Carlo schemes in computing the competitive prices of variance swaps, with maturity ranging from 3 months to 2 years. The level of aggregate demand and inventory is set to be 1,000, and the other parameters are given in Table 2.1. The prices are estimated using 10^6 (500,000 plus 500,000 antithetic) sample paths with a time step equal to 10^{-3}. The associated standard errors (s.e.) are given in parentheses and the 90% confidence intervals (C.I.) in brackets. These numbers have been multiplied by 100 to ease reading. The simulation time (in second) is for CPU 1.8 GHz in a Labtop computer.



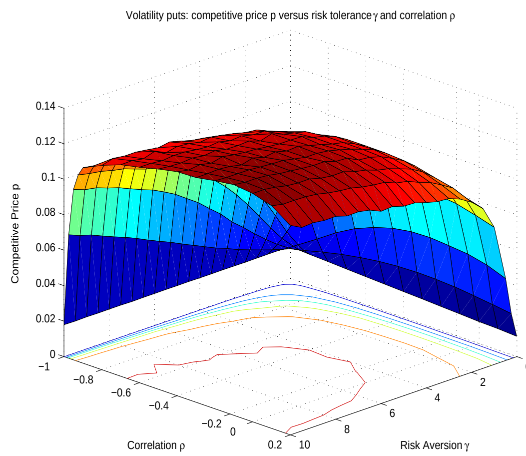
(a) Volatility Calls



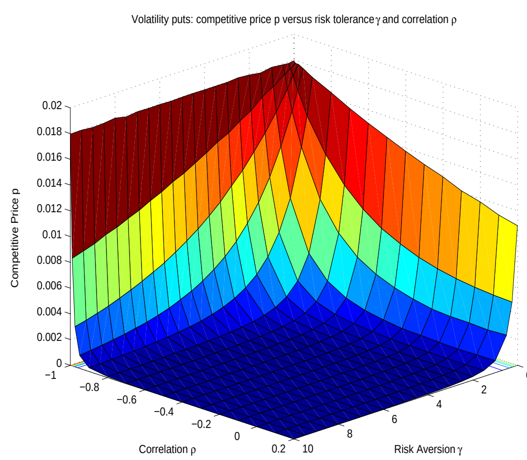
(b) Variance Puts

Figure 2.1: Competitive prices of calls and puts as functions of K and T

In Figure 2.2, we use a European put on volatility as an example to investigate the dependence of the competitive price on the aggregate risk-aversion parameter $\gamma = \frac{1}{\tau}$ and the correlation coefficient ρ. The demand pressure for the graph on the left is positive ($D = 100$), while that on the right is negative ($D = -100$). As we can see, for positive demand pressure, the competitive price is increasing with respect to γ. It is, however, decreasing in γ when the demand pressure is negative.



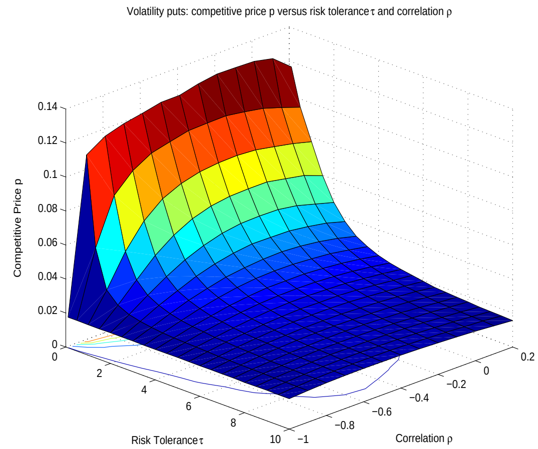
(a) Positive Demand



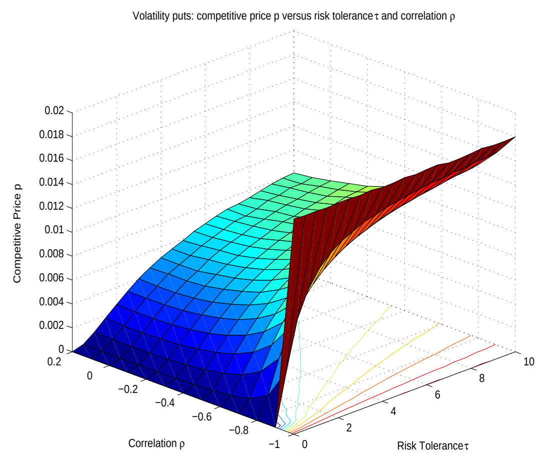
(b) Negative Demand

Figure 2.2: Competitive prices of volatility puts as functions of γ and ρ

Similarly, we plot the competitive price as a function of the aggregate risk-aversion parameter τ and the correlation coefficient ρ, as shown in Figure 2.3. The graph on the left presents a volatility put with positive demand pressure ($D = 100$), while the one on the right has negative demand pressure with $D = -100$. Clearly, we see that for positive demand pressure the competitive price is decreasing in risk tolerance τ, while for negative demand pressure, it is increasing.



(a) Positive Demand



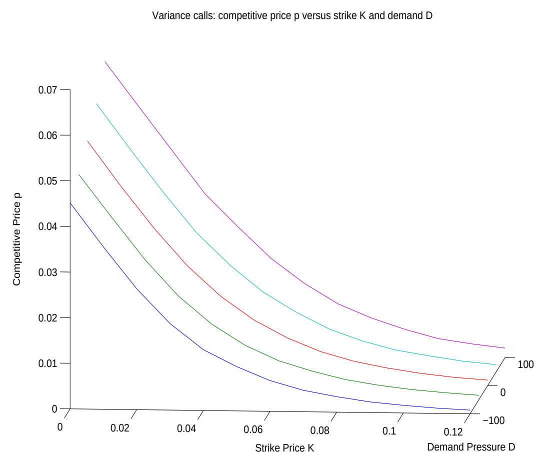
(b) Negative Demand

Figure 2.3: Competitive prices of volatility puts as functions of τ and ρ

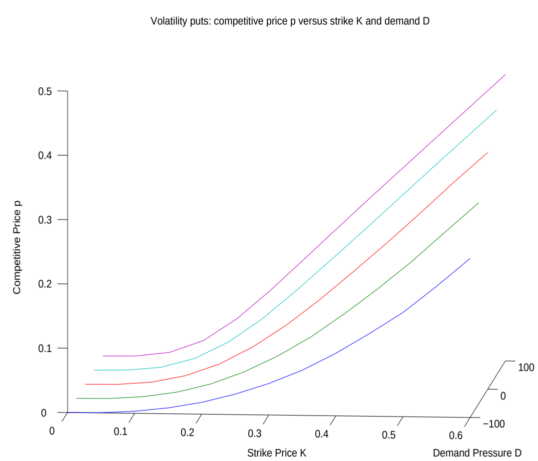
From Figure 2.2 and Figure 2.3, we also observe that, as $\rho \to -1$, the competitive price becomes a constant (independent of γ and τ). This is the consequence of the following result

$$\lim_{|\rho|\to 1} p(G;\rho) = E_{Q_0}\left[g(S_T, Y_T)\middle|Y_t = y, Z_t = z\right].$$

as shown in the previous section.



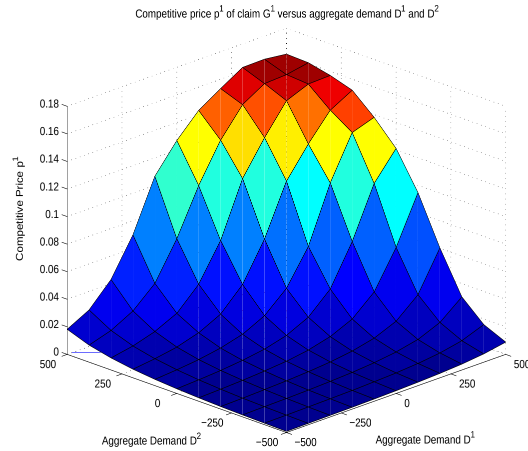
(a) Variance Calls



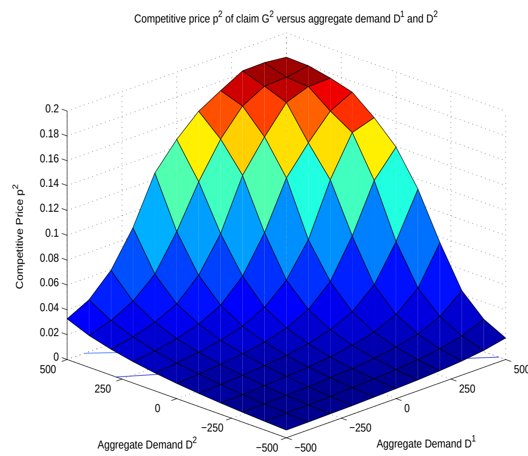
(b) Volatility Puts

Figure 2.4: Competitive prices as functions of strike price K

We next investigate the volatility smile under different levels of demand pressure. Figure 2.4 shows the dependence of the competitive price on the strike prices K for various levels of aggregate demand and inventory pressure. The graph on the left represents a variance call, while the one on the right is a volatility puts. In both cases, the demand pressure D range from -100 to 100. Volatility smiles are clearly shown in both graphs for every level of demand pressure. We also see that the competitive price is increasing with respect to the levels of demand pressure.



(a) Out-of-The-Money Variance Call G^1

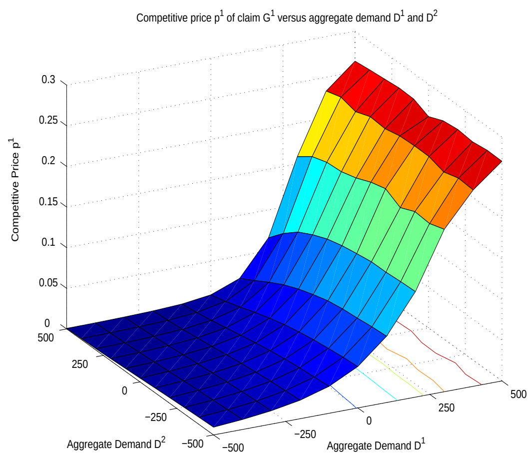


(b) In-The-Money Variance Call G^2

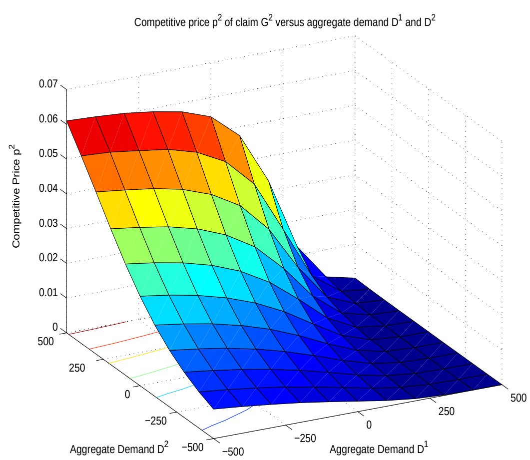
Figure 2.5: Effects of demand and inventory on competitive prices

In what follows, we focus on the effects of demand and inventory on the competitive prices. For simplicity, we consider a market with two volatility derivatives and investigate the cross effects of demands on their prices. Figure 2.5 shows such cross effects for an out-of-the-money (OTM) variance call and an in-the-money

(ITM) variance call. The strike price of the former is $K = 0.05$ and that of the latter is $K = 0.03$. The other parameters are shown in Table 2.1. In this case, the two payoffs are positively correlated and share a similar magnitude. One thus observes a symmetric effect of demands. The competitive price of each claim is increasing as the demand of any claim increases. The demand pressure from either claim has a similar effect on the prices of both claims.



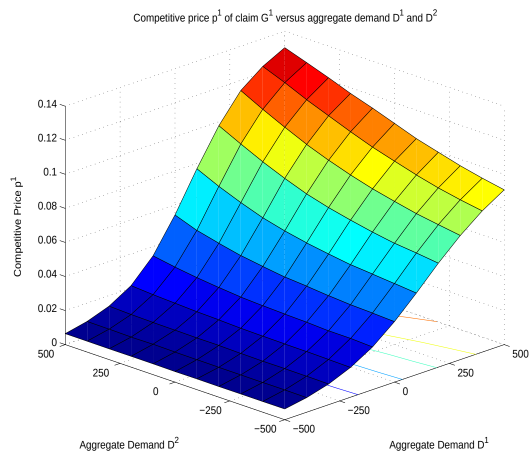
(a) At-The-Money Volatility Call G^1



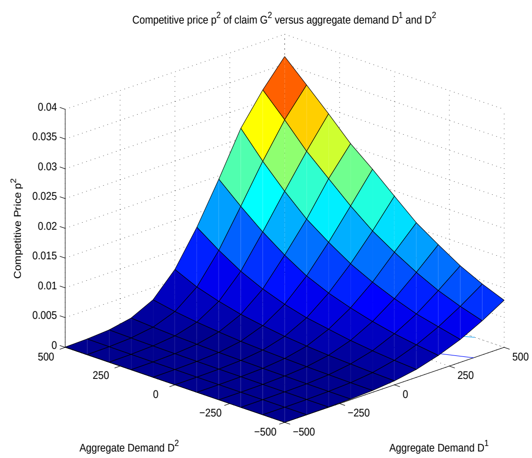
(b) At-The-Money Volatility Put G^2

Figure 2.6: Effects of demand and inventory on competitive prices

In Figure 2.6, we turn to study the cross effects of demands for at-the-money (ATM) volatility call and put, whose strike prices are both $K = 0.2$. In this case, their payoffs are negatively correlated. The price of the ATM call is increasing with respect to its own demand pressure, while decreasing slightly in the demand of the ATM put. The latter effect is insignificant because the payoff of the ATM put is very small comparing to that of the ATM call. On the other hand, the price of the ATM put is decreasing in the demand of the ATM call, though it is increasing in its own demand pressure.



(a) In-The-Money Volatility Put G^1



(b) Out-of-The-Money Volatility Put G^2

Figure 2.7: Effects of demand and inventory on competitive prices

Finally, we investigate the cross effects of demands for in-the-money (ITM) and out-of-the-money (OTM) volatility puts, as Figure 2.7. The strike price of the former is $K = 0.25$ and that of the latter is $K = 0.15$. In this case, the payoff of ITM put is significantly greater than that of the OTM put, although they are positively correlated. We thus observe that the prices of both claims are increasing in the demand of either claim. The price effect from the demand of the ITM put is more significant.

2.6 Conclusions

We have developed a partial equilibrium model to price non-redundant derivatives in incomplete markets. In a Markovian diffusion setting, we established the competitive equilibrium and derived a closed-form pricing formula based on the competition among an arbitrary number of market makers.

Our model further shows that the price of a derivative is increasing with the demand of any derivative in the derivative market. The increasing rate is proportional to the covariance, under the competitive pricing measure, between the unhedgeable parts of the payoffs of the associated derivatives. This result contributes to explain the *options-pricing puzzles*, namely, index options appear to be expensive and low-moneyness options seem to be especially expensive comparing to other individual equity options.

The competitive pricing formula turns out to be the natural generalization of the marginal utility pricing formula. We have also investigated the various properties of the competitive price and its relations to the existing pricing concepts, such as marginal utility price and indifference price. In addition, we applied our model to price a variety of volatility derivatives and developed the associated numerical schemes to compute the competitive prices under Heston stochastic volatility model.

In the future, we plan to extend the model to the semimartingale setting (see Chapter 2). Another possible direction is to impose some constraints (wealth, initial endowment, etc.) to the model.

Chapter 3

Competition-Based Derivatives Pricing: A Semimartingale Framework

3.1 Introduction

Most traditional asset pricing theories rely on the rational optimizing behavior of individuals and their direct interaction through markets, which implies that financial intermediaries play no role in the markets and the determination of security prices [see 47]. Such an extreme view is, however, inconsistent with what is observed in practice: (i) Intermediaries, such as banks and insurance companies, have existed for a long history and been playing a central role in the provision of financial products that create values for their customers; (ii) Average individuals did not have to make complicate financial decisions in the past, are not trained to make in the present, and are unlikely to execute efficiently in the future, even with attempts at education [106];[1] (iii) Over the past several decades, individuals have been shifted dramatically away from direct participation in financial markets towards participation through various intermediaries [see 3]. In short, intermediaries have been of central importance in financial markets, particularly in derivative markets, and it is

[1]Even individuals have the knowledge of investment, active participation in markets is costly or even unrealistic for individuals because, in doing so, they have to devote tremendous time and effort to learning market behaviors and monitoring their portfolios through time.

evident that they will become more and more significant in the future.

This paper attempts to price derivatives from the viewpoint of financial intermediaries, such as banks, insurance companies, options market makers, etc. In terms of Merton's (1993) *functional perspective*,[2] these intermediaries perform their functions of risk transformation and management via "manufacturing/engineering" financial products. Instead of acting as agents, they are active counterparts in derivatives markets, who supply specific derivative securities in principal transactions for the demands of their customers (households and business firms).[3] Like other business firms, intermediaries maximize the interests of their shareholders. Because they create explicit liabilities whenever selling products, intermediaries pursue optimal trading strategies in the capital markets (stocks, bonds, etc.) or contracting with other institutions to manage their risk exposures.[4] Given this, the derivative pricing problem is usually embedded into that of risk management [99].

If a derivative can be replicated costlessly through dynamic trading, the classical Black-Scholes-Merton (BSM) theory applies, and the derivative price is uniquely determined by the no-arbitrage arguments,[5] which is irrelevant to the customers' demand. However, such a derivative is a perfectly redundant security that adds nothing new to the market and no social welfare, as argued by [66] in his *Catch 22 on Option-Pricing*. In reality, a variety of non-redundant products are desirable, because households and business firms bear various "basis" risks, such as human capital, credit, weather, mortality, and catastrophe.[6] To accommodate such demands, [105] suggested some innovative products for the future financial practice. One common issue arising in the innovation process is that these prod-

[2] The functional perspective of financial intermediation is based on the economic functions performed by the intermediaries, such as risk transformation, rather than the activities of existing institutions as in the traditional institutional approach. For elaboration, we refer to a series of papers by [98, 100, 102, 101, 104, 105]. See also [123, 30] and [106].

[3] Such transactions can take place either in organized options exchanges or in over-the-counter (OTC) markets. In the former, intermediaries usually act as dealers who provide liquidity to impatient traders, while in the latter, they offer a variety of specific (tailored) products that cannot be efficiently supported by direct trading in organized markets.

[4] In practice, the production process includes an activity of production identification, which may involve issues relevant to security design. For that concern, we refer to, among others, [8, 45], and [93].

[5] See [14] and [97]. Also, [103] provided a detailed descriptions on the production process in practice that applies option-pricing theory.

[6] Under the "financial innovation spiral", it is such demand by customers for non-redundant derivatives that create/drive the business of intermediaries and further stimulate the development of financial industry.

ucts cannot be efficiently hedged using financial markets. In addition, even the exchange-traded options are not perfectly replicable in the real world because of various market imperfections, such as transaction costs, stochastic volatility, and unpredictable jumps [48].

In this paper, we consider an incomplete semimartingale market consisting of arbitrary numbers of fundamental securities and non-redundant derivatives. Our goal herein is to determine the prices of these derivatives given their contractual payoffs. We assume an arbitrary number of risk-averse intermediaries in the economy, who compete to provide the derivatives for the demand of their customers and trade dynamically in the primary market to hedge part of their net risk exposures. They manage these two activities optimally in terms of utility maximization. In equilibrium, the derivative prices are in turn determined by imposing the market-clearing condition. It is not our intention to address the various incentives of derivative demand by every individual customer. We, instead, incorporate directly the aggregate demand pressure into the classical competitive equilibrium model. This simplification allows us to analyze explicitly the price impact of the demand pressure.

The competitive assumption is justified by the growing competition within the financial industry and from outside. Technology advances, particularly the growth of the internet, have been cutting down the entry costs and driving the financial intermediation towards a more competitive industry. Meanwhile, the emergence of aggregators, such as LendingTree and Progressive, allows more effective competition since customers can compare a wide base of potential suppliers [see 25]. Moreover, intermediaries have to face the competition from outside as many big brand non-financial firms, such as GM and Sony, have started offering financial products. In addition, it is evident that many exchange markets, such as NASDAQ and London Stock Exchange, feature competition between market makers. As noted by [134], market makers such as specialists in New York Stock Exchange (NYSE) typically face competition from floor traders, competing dealers, limit orders, and other exchanges.[7] In market microstructure, models of competition among market makers have been developed by [70] and others.

The motivation to incorporate the demand effects into our equilibrium model

[7] Limit orders are orders to buy (sell) that specify a maximum (minimum) price at which the trader is willing to transact. The traders who issue limit orders are indeed acting as dealers, though they often do not recognize this. A market order is an order to buy (sell) at prevailing prices. The traders that submits market order can be viewed as customers. For more discussion, we refer to [67].

lies mainly in the fact that, because of various constraints, intermediaries are sensitive to unhedgeable risks [128]. Since they cannot hedge perfectly, intermediaries have to bear significant inventory risks, which apparently depend on their customers' specific demand. In practice, intermediaries usually adjust their prices for selected products so as to induce their customers to buy products that reduce the overall pressure of hedging [101, 124]. Consequently, such demand pressure becomes an important factor for the determination of derivative prices. Similar approach is adopted in many security exchanges, where market makers (dealers) actively adjust the price quotations in response to fluctuations in their inventory levels [see 130, 57]. Based on such observations, the inventory control has become a standard approach in market microstructure; see, for example, [133, 4, 94], and [95]. Moreover, recent empirical research has found a strong correlation between the options prices (implied volatilities) and the net buying pressure [see, for example, 15]. Such correlation provides a possible resolution to the so-called *option-pricing puzzles*, which will be further discussed later on. Motivated by such intensive evidence, we model the price impact of demand pressure by incorporating the customers' aggregate demand into the competitive equilibrium model.

Our pricing methodology expands the economic insights of utility-based approach that has been studied extensively in mathematical finance. One popular valuation concept is the *indifference pricing*, in which the derivative price is defined as the amount that compensates an optimally behaving investor (representative agent) for taking the risk to transact the derivative. In other words, the indifference buying (resp. selling) price would make the (representative) investor indifferent between buying (resp. selling) the derivative and disregarding such a trading opportunity. This pricing concept was introduced by [72] and has received a great attention recently. We refer to, among others, [118, 40, 111], and [83]. It is worth noting that the indifference prices are essentially the reservation prices of traders. They define a range in which trading can take place, namely, the buying price is the highest one that the buyer is willing to buy the derivative, while the selling price is the lowest one that the writer is willing to sell. However, as shown in [129], the indifference selling price is always higher than the buying price, which implies that such pricing criterion does not allow an equilibrium in the derivative market; see also [35].

Alternatively, [32] introduced the concept of *marginal utility pricing* ("fair price") based on the idea of "marginal rate of substitution". Precisely, the fair price is defined to be such that the *representative agent* cannot increase her/his expected utility by diverting an infinitesimal amount of his/her capital into the claim. Unlike indifference prices, the marginal price is linear, symmetric for buyer and seller, and it turns out to be the limit of indifference price for an infinitesimal quantity. This pricing concept has been further studied by, among others, [79, 80] and [51]. The uniqueness of the marginal price was proved by [74]. More recently, [85, 86] analyzed the sensitivity and asymptotic property of the marginal price.

To the best of our knowledge, most of these models are based on the assumption that there is a single (representative) agent who has the power to determine the prices. Also commonly assumed is that the derivatives have zero demand and inventory pressure, which presumably erases their effects on the prices. Moreover, despite the *ad hoc* pricing criteria and technical advances, a satisfactory understanding of the economic nature of utility-based pricing and its ultimate alignment with classical asset equilibrium concepts are still lacking in the literature. In contrast, we model directly the competition among an arbitrary number of financial intermediaries based on their optimal behavior in an incomplete semimartingale market, and analyze the price effects of the aggregate demands by customers. Our approach incorporates the effects of demand and inventory into the classical concept of equilibrium and produce a meaningful pricing formula.

Besides the contribution to the modelling of intermediation, our work provide insights to explain the *option-pricing puzzles* – that index options appear to be expensive and that low-moneyness options seems to be especially expensive. These puzzles have been well documented in the literature of empirical options pricing; see, among others, [91, 9, 28, 16, 5], and [26].

One possible resolution was initiated by [15], who studied the S&P 500 index options (SPX) market and found that option implied volatilities are positively correlated to the net buying pressure for options. They further used the net buying effect to explain the shape of the volatility "smile" or "smirk" across different option series. Many others followed this approach. For example, [21, 22] examined the net buying effects using the Hong Kong Hang Seng index options and supported Bollen and Whaley's results. More recently, [56] further analyzed the demand effects on option prices in a discrete time market with a single agent and documented that customers tend to have a net long SPX option and a short equity option position. They concluded that demand pressure help to explain the relative expensiveness of index options. Consistently, our model shows the price of a derivative is increasing with the demand of another derivative in a rate proportional to the covariance, under the competitive pricing measure, between the unhedgeable parts of the two associated derivative payoffs. This result helps to explain the options-pricing puzzles because it is evident that the demands of index options and deep out-of-the-money (OTM) puts are very high comparing to other individual equity options.

In addition, our work relates to optimal positioning in derivatives, see, for example, [20, 19, 90], and [76, 77]. Another stream of literature related to our approach is the market completion, see, for example, [73] and [23].

The remainder of this paper is organized as follows. In Section 3.2, we describe the competition-based pricing model in an incomplete semimartingale market. Section 3.3 provides techniques based on convex duality for solving the underlying optimizing problems and the equilibrium. We derive the competition-based pricing formula and investigate the associated properties in Section 3.4. Section 3.5 focuses on the sensitivity analysis on the price impact of demand. In Section 3.6, we discuss the relations to existing pricing concepts. We further present an example under Markovian diffusion framework in Section 3.7. Finally, Sections 3.8 concludes.

3.2 The Competition-Based Pricing Model

3.2.1 The Market

We fix a finite horizon $T < \infty$ and consider a continuous-time financial market consisting of $d+1$ fundamental securities, one riskless bond S^0 and d ($d \geq 1$) risky stocks $(S^1, \cdots, S^d)$, whose prices are given exogenously. Without loss of generality, the price process of the bond is normalized to one, namely, $S_t^0 \equiv 1$. The price process $S := (S_t^1, \cdots, S_t^d)_{0 \leq t \leq T}$ of the stocks is modelled as a d-dimensional locally bounded semimartingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual conditions of right continuity and completeness, so that all semimartingales are chosen to be right-continuous with left limits (càdlàg).

To exclude arbitrage opportunities among the fundamental securities, we assume the existence of *equivalent local martingale measures* (ELMM), namely, the set of ELMM

$$\mathbb{P}_e := \{Q \sim P | S \text{ is a local } (\mathbb{F}, Q)\text{-martingale}\} \neq \varnothing. \tag{3.1}$$

The equivalence between the absence of arbitrage opportunities and the existence of ELMM has been well established in mathematical finance; see [41, 42] for the precise statements.

We denote by (ϑ, x) a self-financing portfolio with initial capital x and ϑ being an admissible trading strategy that will be defined rigorously later on (see Definition 3.3.1). Then the corresponding *wealth* process $X^{\vartheta,x}$ evolves in time according to the stochastic integral of ϑ with respect to S, namely

$$X_s^{\vartheta,x} := x + (\vartheta \cdot S)_s = x + \int_t^s \vartheta_u \cdot dS_u, \quad t \leq s \leq T. \tag{3.2}$$

Besides the fundamental securities, there are also n ($n \geq 1$) derivatives (contingent claims) available in the market, which are characterized in terms of their terminal payoff at maturity T. The derivative payoffs are given exogenously by n random variables $(G^1, \cdots, G^n)$ that are $\mathcal{F}_T$-measurable. To avoid triviality, we assume that all these derivatives are non-redundant in the sense that any linear combination of G is not replicable by an admissible trading strategy in the underlyings, namely

$$\exists\, \vartheta \in \Theta \quad (\vartheta \cdot S)_T + \alpha \cdot G = \text{const} \quad P\text{-a.s.} \quad \Longrightarrow \quad \alpha = 0. \tag{3.3}$$

It is worth noting that the set $\{E^Q[\alpha \cdot G] \,|\, Q \in \mathbb{P}_e\}$ containing all no-arbitrage prices of the claim $\alpha \cdot G$ is either a unique point or an open interval, the former of which implies that the claim $\alpha \cdot G$ is replicable [see 122, Theorem 5.3]. The above assumption thus means that the set of no-arbitrage prices is an open interval.

In this paper we are interested in the initial price $p_0 := (p_0^1, \cdots, p_0^n)$ or even the entire price process $p_t := (p_t^1, \cdots, p_t^n)$ of the derivatives $G := (G^1, \cdots, G^n)$. For technical reasons we assume that G is bounded. Although some derivatives such as call options have an unbounded payoff, it will not affect the practical accuracy to truncate the payoff conditional on sufficiently extreme events.

3.2.2 The Market Players

As discussed previously, our economy is populated with an arbitrary number of intermediaries as well as their customers. For various internal reasons, such as agency problem, capital constraints, portfolio insurance, or gambling entertainment, customers demand to trade derivatives with intermediaries. We do not intend to address such incentives in details. Instead, we assume that the customers' aggregate demand of derivatives at time t, $D_t := (D_t^1, \cdots, D_t^n)$, is given exogenously. It is worth noting that such demand D_t is observable directly from the market data; see, for example, [15].

On the supply side, intermediaries satisfy the aggregate demand of derivatives through competition. They provide liquidity to the derivatives market while bearing some inventory risk. We assume that at time t there exist a finite number of intermediaries indexed by I_t, who have *constant absolute risk aversion* (CARA). That is, they have exponential utility functions

$$U_i(x) = -e^{-\gamma_i x}, \quad x \in \mathbb{R}, \tag{3.4}$$

with $\gamma_i > 0$ being the risk aversion parameter of intermediary $i \in I_t$. We shall also use the risk tolerance parameter $\tau_i = \frac{1}{\gamma_i}$ because it provides more intuition in interpreting our pricing formula. The objective of intermediaries is to maximize the expected utility of terminal wealth by choosing optimally the (static) derivative positions and a subsequent dynamic hedging strategy.

Now suppose that, with initial endowment of x_i at time t, each intermediary i takes static positions $q_i := (q_i^1, \cdots, q_i^n)$ in the derivatives G. Since the market is competitive, the price vector $p_t = (p_t^1, \cdots, p_t^n)$ of the derivatives is taken as given, and the derivative positions thus cost $(q_i \cdot p_t)$ with "$\cdot$" being the usual inner product on $\mathbb{R}^n$. The intermediary i then invests the remaining capital of $(x_i - q_i \cdot p)$ into the underlyings using an optimal trading strategy ϑ_i, whilst collecting a random derivative payoff $q_i \cdot G$ at maturity T. At time t, each intermediary i wants to maximize the expected utility of terminal wealth over both the static derivative positions q_i and the dynamic trading strategy ϑ_i in the underlyings, and therefore solves the following optimal investment problem:[8]

$$\underset{\vartheta_i \in \Theta, q_i \in \mathbb{R}^n}{\text{maximize}} \; E\left[U_i\left(X_T^{\vartheta_i, x_i - q_i \cdot p_t} + q_i \cdot G\right) \Big| \mathcal{F}_t\right], \tag{3.5}$$

where Θ is the set of admissible trading strategies to be defined in the sequel, and $i \in I_t$. Since we are interested in the equilibrium prices, we assume the existence of the optimal investment strategies and use them to define the competition-based derivative prices. The technical assumptions to ensure such existence will be specify in Section 3.3.

3.2.3 The Competitive Equilibrium

The derivative prices are determined through a competitive equilibrium among the intermediaries as well as their customers. Precisely we introduce the following definition.

Definition 3.2.1. *The competitive price of the derivatives G is defined as vector $p(G, D)$ such that each intermediary i admits an optimal position q_i^* in the derivatives and the derivative market clears. That is, the derivative price vector p satisfies the partial market clearing condition*

$$\sum_{i \in I_t} q_i^*(p) + D_t = 0, \tag{3.6}$$

where q_i^* is the solution to the optimal investment problem (3.5) for intermediary $i \in I_t$.

In the above definition, the derivative prices are set based on the partial equilibrium arguments because no market clearing condition is imposed on the underlyings whose prices are given exogenously. It is a compromise between the no-arbitrage arguments in the classical derivative pricing and the general equilibrium theory in asset pricing. Although it relies on the classical pricing principle of equilibrium from economic theory, this pricing notion is relatively new in derivatives

[8]This optimization problem was previously considered by [77] in the context of indifference pricing. With appropriate assumptions on the price vector p_t, they proved the existence and uniqueness of the optimal solution using an approach relying heavily on the properties of indifference price. We will offer a straightforward approach to solve such optimization problem in Section 3.4

pricing, in that it introduces competition and demand effect into the pricing mechanism. Some similar definitions based on partial equilibrium can be found in, for example, [73, 23], and [56].

The difficulty in solving the equilibrium partially lies in the fact that the optimization problem (3.5) in general does not admit a closed-form solution. However, it is useful to observe that the optimization problem can be written as

$$\max_{q_i \in \mathbb{R}^n} u\left(\gamma_i(x_i - q_i \cdot p), \gamma_i q_i; G\right), \tag{3.7}$$

where we have introduced the value function

$$u(x, \alpha; G) := \sup_{\vartheta \in \Theta} E\left[-\exp\left\{-\left(x + \int_t^T \vartheta \cdot dS + \alpha \cdot G\right)\right\}\Big|\mathcal{F}_t\right], \tag{3.8}$$

with Θ being the set of admissible trading strategies to be defined in the sequel.

For fixed G and α, the value function u measures the maximal utility of terminal wealth arising from the derivative payoff $\alpha \cdot G$ together with the value of an optimal trading strategy starting with initial capital x. Thanks to the fact that Θ is a cone, we choose to absorb the risk aversion parameter into other parameters when writing the value function. This is tailored to emphasize the essential structure of the problem and ease our subsequent presentation. The valuation problem (3.8) and its variations have been studied extensively in mathematical finance. Convex duality is one of the most popular approaches because the dual problem provides a simpler structure and admits a semi-closed form solution; see, among others, [54, 40, 78], and [10].

3.3 Utility Maximization and Convex Duality

In this section we prepare the technical ground for solving the equilibrium model. The key component is the convex duality, in particular the dual relation between the exponential utility maximization problem (3.8) and the associated entropy minimization problem

$$v(\alpha; G) = \inf_{Q \in \mathbb{P}_f} \left(H(Q|P) + E^Q[\alpha \cdot G]\right), \tag{3.9}$$

where $\mathbb{P}_f$ is a set of local martingale measures to be defined below.

We recall that the *relative entropy* $H(Q|P)$ of any probability measure Q with respect to P is defined as

$$H(Q|P) := \begin{cases} E\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\middle|\mathcal{F}_t\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P. It is worth noting that the relative entropy $H(Q|P)$ measures the distance between the two probability distributions Q and P, which is always nonnegative [see 75]. The set $\mathbb{P}_a$ of *absolutely continuous local martingale measures* (ALMM) for S with respect to $\mathbb{F}$, and the set $\mathbb{P}_f(P)$ of ALMM with finite entropy are defined as

$$\mathbb{P}_a := \{Q \ll P | S \text{ is a local } (\mathbb{F}, Q)\text{-martingale}\},$$
$$\mathbb{P}_f := \mathbb{P}_f(P) := \{Q \in \mathbb{P}_a | H(Q|P) < +\infty\}.$$

To proceed, we define a probability measure P_α through density

$$\frac{dP_\alpha}{dP} = \frac{1}{E\left[e^{-\alpha\cdot G}\right]}e^{-\alpha\cdot G}, \tag{3.10}$$

for $\alpha \in \mathbb{R}^n$. This is sometimes called an Esscher transform or Esscher change of measure [see 60]. In the sequel, it will be often convenient to work with measure P_α rather than P. Since G is bounded, it is obviously that $P_\alpha \sim P$ and we can write

$$H(Q|P_\alpha) = H(Q|P) + E^Q[\alpha \cdot G] + \log E\left[e^{-\alpha\cdot G}\right], \tag{3.11}$$

which implies that $H(Q|P_\alpha) < +\infty$ if and only if $H(Q|P) < +\infty$. Therefore,

$$\mathbb{P}_f(P_\alpha) = \mathbb{P}_f(P),$$

and we simply write $\mathbb{P}_f$ hereafter. In addition, since P_α is equivalent to P, we could obviously replace P by P_α in the definitions of $\mathbb{P}_a$ and $\mathbb{P}_e$ without altering them.

To this end, we are ready to specify the set of admissible trading strategies Θ.

Definition 3.3.1. *A self-financing trading strategy ϑ is called admissible if it is*

$\mathbb{F}$-predictable S-integrable $\mathbb{R}^d$-valued process such that the stochastic integral $\vartheta \cdot S$ (namely, the wealth process $X^{\vartheta,x}$) is a $(\mathbb{F}, Q)$-martingale for all $Q \in \mathbb{P}_f$. We denote by Θ the set of all admissible trading strategies, namely

$$\Theta := \{\vartheta \in L(S) | \vartheta \cdot S \textit{ is a } (\mathbb{F}, Q)\textit{-martingale for all } Q \in \mathbb{P}_f\}.$$

We remark here that there are actually many choices for Θ that leads to the same value function u, namely, the same dual problem; see [40] for some examples. Generally speaking, it is enough to ensure that Θ contains the optimal trading strategy whose value is given by the dual problem. Technically, we require that Θ satisfies the robust duality, namely

$$\begin{aligned} &\sup_{\Theta} E\left[-\exp\left\{-\left(\int_t^T \vartheta \cdot dS + \alpha \cdot G\right)\right\}\right] \\ &= -\exp\left\{-\inf_{Q \in \mathbb{P}_a} \left(H(Q|P) + E^Q[\alpha \cdot G]\right)\right\}. \end{aligned} \tag{3.12}$$

We adopt the above definition for intuitive sake, though other choices of Θ will not affect our competition-based price since its definition is based on u.

To obtain the duality results, we need to assume that there exists some ELMM with finite entropy, i.e.

$$\mathbb{P}_f \cap \mathbb{P}_e \neq \varnothing, \tag{3.13}$$

where $\mathbb{P}_e$ is the set of ELMM defined in (3.1). Under assumption (3.13), it has been shown that there is a unique martingale measure $Q(\alpha) \in \mathbb{P}_f \cap \mathbb{P}_e$ minimizing the relative entropy $H(Q|P_\alpha)$ with respect to P_α over all $Q \in \mathbb{P}_f$. Moreover, the density of $Q(\alpha)$ is given by

$$\frac{dQ(\alpha)}{dP} = \frac{dQ(\alpha)}{dP_\alpha}\frac{dP_\alpha}{dP} = \exp\left\{c_\alpha - \int_0^T \vartheta_u^\alpha \cdot dS_u - \alpha \cdot G\right\}, \tag{3.14}$$

for some $\vartheta^\alpha \in L(S)$ such that $\vartheta^\alpha \cdot S$ is a $Q(\alpha)$-martingale and $c_\alpha \in \mathbb{R}$; see Theorem 2.2 and 2.3 of [**?**]Frittelli00MF and Proposition 3.2 and the proof of Theorem

4.3 [64]. The identity (3.11) further shows that

$$\inf_{Q\in\mathbb{P}_f} H(Q|P_\alpha) - \log E\left[e^{-\alpha\cdot G}\right] = \inf_{Q\in\mathbb{P}_f}\left(H(Q|P) + E^Q[\alpha\cdot G]\right)$$

and both infima are attained by the same minima $Q(\alpha)$. In terms of $Q(\alpha)$, we thus write

$$v(\alpha; G) = H(Q(\alpha)|P) + E^{Q(\alpha)}[\alpha\cdot G]. \tag{3.15}$$

It is worth noting that for fixed G, the minimal entropy martingale measure $Q(\alpha)$ depends only on the parameter α. We write it as a function of α for our subsequent purpose.

Now we are ready to present the duality results on the exponential utility maximization problem and the associated dual problem, which is due to [40] and [78].

Proposition 3.3.2. *Assume that the price process S is locally bounded and admits an equivalent local martingale measure with finite entropy. Then the utility maximization problem (3.8) admits the following dual relation*

$$\begin{aligned}&\sup_{\vartheta\in\Theta} E\left[-\exp\left\{-\left(x + (\vartheta\cdot S)_T + \alpha\cdot G\right)\right\}\right]\\&= -\exp\left\{-\inf_{Q\in\mathbb{P}_f}\left(H(Q|P) + E^Q[\alpha\cdot G]\right) - x\right\}.\end{aligned} \tag{3.16}$$

Moreover, the supremum in the primer problem is attained by some strategy $\vartheta^\alpha\in\Theta$ and the infimum in the dual problem is attained by a unique martingale measure $Q(\alpha)\in\mathbb{P}_f\cap\mathbb{P}_e$ with density defined in (3.14).

The above result identifies the duality relation between the maximization of expected exponential utility with contingent claims over a space of admissible trading strategies and the minimization of relative entropy with a correction term involving the claims over a space of martingale measures. This duality result enables us to study the utility maximization problem through its dual problem because the latter provides a simpler structure. In the sequel, we investigate some important properties of the dual problem (3.9) that will be useful for solving our equilibrium.

To facilitate the subsequent presentation, we define a functional

$$f(Q;\alpha) := \left(E^{Q}[\alpha \cdot G] + H(Q|P)\right), \tag{3.17}$$

and rewrite the dual problem (3.9) as

$$v(\alpha;G) = \inf_{Q \in \mathbb{P}_f} f(Q;\alpha) = f(Q(\alpha);\alpha). \tag{3.18}$$

In addition, we will frequently use Csiszár's theorem [31] on minimal entropy measures. For ease of future reference, we state it as a lemma in our framework:

Lemma 3.3.3. *A measure $R \in \mathbb{P}_f$ minimizes entropy $H(Q|P)$ over $Q \in \mathbb{P}_a$ (or $\mathbb{P}_f$) if and only if*

$$H(Q|P) \geq H(Q|R) + H(R|P) \tag{3.19}$$

for every $Q \in \mathbb{P}_a$ (or $\mathbb{P}_f$).

To this end, from (3.14), it can be shown that $c_\alpha = v(\alpha;G)$ by multiplying dP_α/dP and taking log and expectation with respect to $Q(\alpha)$ on the both sides. Thus, plugging into (3.14), the density of $Q(\alpha)$ becomes

$$\xi(\alpha) := \frac{dQ(\alpha)}{dP} = \exp\left\{v(\alpha;G) - \int_0^T \vartheta_u^\alpha \cdot dS_u - \alpha \cdot G\right\}, \tag{3.20}$$

for some $\vartheta^\alpha \in L(S)$ such that $\vartheta^\alpha \cdot S$ is a $Q(\alpha)$-martingale.

Regularity. The following theorem summarizes the regularity results related to the dual problem, which will be useful for solving the optimal investment problems.

Theorem 3.3.4.
(i) The value function $\alpha \mapsto v(\alpha;G)$ is (absolutely) continuous on $\mathbb{R}^n$.
(ii) The functions $\alpha \mapsto E^{Q(\alpha)}[G]$ and $\alpha \mapsto H(Q(\alpha)|P)$ are continuous on $\mathbb{R}^n$.
(iii) Moreover, the function $\alpha \mapsto v(\alpha;G)$ is continuously differentiable on $\mathbb{R}^n$ and its gradient is given by

$$\nabla v(\alpha;G) = E^{Q(\alpha)}[G], \tag{3.21}$$

with $Q(\alpha)$ being defined in (3.20).

Proof. We provide a detail proof for the above results in Appendix A.1. $\square$

It is worth noting that the above results can be easily apply to establish the differentiability of indifference price using its well-known representation formula. A different approach in studying such differentiability can be found in Theorem 5.1 of [77], where they relies on the related representation and asymptotic results on indifference prices. The approach we used in Theorem 3.3.4 are actually more general and straight for handling the regularity of such a value function v. It indicates that such regularity is directly implied by the uniform boundness of $E^Q[G]$.

Concavity. We next consider the concavity of v.

Theorem 3.3.5. *The value function $v(\alpha; G)$ is strictly concave in α on $\mathbb{R}^n$.*

Proof. We first fix a measure $Q \in \mathbb{P}_f \cap \mathbb{P}_e$, and it is clear that the function

$$f(\alpha; G) = \alpha \cdot E^Q[G|\mathcal{F}_t] + H(Q|P)$$

is affine in α. Thus, from (3.9) we see that $v(\alpha; G)$ is an infimum of affine functions on $\mathbb{R}^n$, and hence is concave. We next argue that such concavity is indeed strict. Otherwise, let us suppose that it is not true, then there must be some $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha \neq \beta$ and v is affine on the line segment between α and β. From the identity (3.11), we find that

$$\begin{aligned}\big(H(Q(\beta)|P_\alpha) - H(Q(\alpha)|P_\alpha)\big) + \big(H(Q(\alpha)|P_\beta) - H(Q(\beta)|P_\beta)\big)\\ = (\alpha - \beta) \cdot \left(E^{Q(\beta)}[G] - E^{Q(\alpha)}[G]\right).\end{aligned}$$

The right-hand side equals zero due to our assumption that v is affine on the line segment and the gradient formula (3.21). The left hand side is a sum of two nonnegative terms, hence both are zero. This implies $Q(\beta) = Q(\alpha)$ by the uniqueness of the minimal entropy martingale measure. Thus, applying (3.14) to $Q(\beta)$ and $Q(\alpha)$ shows that

$$(\vartheta^\beta - \vartheta^\alpha) \cdot S + (\beta - \alpha) \cdot G = c_\beta - c_\alpha = \text{const.}$$

It further follows from our assumption (3.3) that $\beta - \alpha = 0$, which contradicts to $\alpha \neq \beta$ and completes the proof. $\square$

3.4 The Competition-Based Derivative Prices

In this section, we devote to construct the competition-based derivative prices $p(G, D)$. We start from solving the optimal investment problems (3.8). The following theorem provides the existence and uniqueness for the optimal investment problem and characterize the solutions in terms of the first-order condition.

Theorem 3.4.1. *The intermediaries' optimal investment problems (3.7) admit a solution (ϑ_i, q_i) and the optimal derivative positions q_i is unique provided that the derivatives are non-redundant. Moreover, the optimal derivative positions q_i are characterized by the first-order condition*

$$E^{Q(\gamma_i q_i)}\left[G|\mathcal{F}_t\right] - p_t = 0, \tag{3.22}$$

where $Q(\alpha) \in \mathbb{P}_f \cap \mathbb{P}_e$ is the minimal entropy martingale measure defined in (3.20).

Proof. The existence of ϑ_i have been well established in the duality arguments. Concerning the optimal derivative position q_i, we first combine (3.8), (3.9), and (3.16) to write

$$u(x, \alpha; G) = -\exp\left\{-(x + v(\alpha; G))\right\}, \tag{3.23}$$

where v is the value function of the dual problem given by (3.15). We thus observe that

$$\max_{q_i \in \mathbb{R}^n} u\left(\gamma_i(x_i - q_i \cdot p), \gamma_i q_i; G\right) = -e^{-\gamma_i x_i} \exp\left\{- \max_{q_i \in \mathbb{R}^n} \left(v(\gamma_i q_i; G) - \gamma_i q_i \cdot p\right)\right\}.$$

Thus, the existence and uniqueness of q_i follow directly from those of $v(\alpha; G)$ as a function of α. To see the existence, we fix a measure $Q \in \mathbb{P}_f \cap \mathbb{P}_e$, and it is clear that the function

$$f(\alpha; G) = \alpha \cdot E^Q[G|\mathcal{F}_t] + H(Q|P)$$

is affine in α. Thus, from (3.9) we see that $v(\alpha; G)$ is an infimum of affine functions on $\mathbb{R}^n$, and hence is concave. This proves the existence.

For the uniqueness, let us assume that $\alpha, \beta \in \mathbb{R}^n$ are two maxima of v, then we have $v(\alpha; G) = v(\beta; G)$ and $E^{Q(\alpha)}[G] = E^{Q(\beta)}[G] = 0$. From (A.2), we find

$$H(Q(\beta)|P_\alpha) - H(Q(\alpha)|P_\alpha) = v(\alpha; G) - v(\beta; G) - (\beta - \alpha) \cdot E^{Q(\beta)}[G] = 0,$$

which implies that $Q(\beta) = Q(\alpha)$ since the minimal entropy martingale measure is unique. Thus, applying (3.14) to $Q(\beta)$ and $Q(\alpha)$ shows that

$$(\vartheta^\beta - \vartheta^\alpha) \cdot S + (\beta - \alpha) \cdot G = c_\beta - c_\alpha = \text{const.}$$

It further follows from our assumption (3.3) that $\beta - \alpha = 0$, which proves the uniqueness.

The first-order condition of u is given by

$$\begin{aligned} 0 &= \nabla_{q_i} u\left(\gamma_i(x_i - q_i \cdot p), \gamma_i q_i; G\right) \\ &= \gamma_i u\left[p - \nabla_\alpha v(\gamma_i q_i; G)\right], \end{aligned}$$

which yields that

$$p - \nabla_\alpha v(\gamma_i q_i; G) = 0. \tag{3.24}$$

Thus the first order condition (3.22) follows by applying the gradient formula (3.21). □

Remark 3.4.2. *The existence and uniqueness of q_i are implied by Theorem 3.3.5. We provide such a proof in Appendix A.1.*

In the next theorem, we summarize the main results on the pricing formula and the associated pricing measure.

Theorem 3.4.3.
(i) Given any demand D from the customers, there exists a unique equilibrium for the derivatives market.
(ii) Let τ denote the aggregate risk tolerance parameter of the intermediaries in the economy, namely, $\tau := \sum_{i \in I_t} \tau_i = \sum_{i \in I_t} \frac{1}{\gamma_i}$, and define the probability measure

$P_* \sim P$ through the Esscher transform

$$\frac{dP_*}{dP} = \exp\left\{\frac{1}{\tau} D \cdot G\right\} \Big/ E\left[\exp\left\{\frac{1}{\tau} D \cdot G\right\}\right], \tag{3.25}$$

where D is the aggregate demand of the derivatives G. Then the competition-based derivative prices $p(G, D)$ is given by the condition expectation

$$p_t = E^{Q_*}\left[G|\mathcal{F}_t\right], \tag{3.26}$$

with Q_* being the minimal entropy martingale measure with respect to the prior P_*, namely, $Q_* = \arg\min_{Q \in \mathbb{P}_f} H(Q|P_*) = \arg\min_{Q \in \mathbb{P}_f} H(Q|P) - \frac{1}{\tau} E^Q[D \cdot G]$.
(iii) Moreover, the pricing measure $Q_* \in \mathbb{P}_f \cap \mathbb{P}_e$ is unique and given by density

$$\frac{dQ_*}{dP} = \exp\left\{c_* - \frac{1}{\tau}\left(\int_0^T \vartheta_u^* \cdot dS_u - D \cdot G\right)\right\}, \tag{3.27}$$

where $c_* = v(-D/\tau; G)$ is a constant and $\vartheta^* \in \Theta$ attains the supremum of the primer problem:

$$\sup_{\vartheta \in \Theta} E\left[-\exp\left\{-\frac{1}{\tau}\left(X_T^{\vartheta,x} - D \cdot G\right)\right\}\right]. \tag{3.28}$$

Proof. We shall prove the existence and uniqueness by constructing explicitly the equilibrium. From Theorem 3.4.3, we see that for any given price vector p_t, there exists a unique solution α_p to the equation

$$E^{Q(\alpha)}[G|\mathcal{F}_t] - p_t = 0,$$

where $Q(\alpha)$ is given by (3.20). Thus the first-order condition (3.22) shows that the optimal derivative positions q_i of the intermediary i satisfy $\gamma_i q_i = \alpha_p$, i.e.

$$q_i = \frac{\alpha_p}{\gamma_i} = \alpha_p \tau_i.$$

Combining the market clearing condition (3.6) yields that

$$\alpha_p = -\frac{D}{\sum_{i \in I_t} \tau_i} = -\frac{D}{\tau}.$$

To this end, the first-order condition (3.22) again shows that the unique equilibrium price p_t is given by

$$p_t = E^{Q(-D/\tau)}[G|\mathcal{F}_t].$$

Therefore, we conclude by denoting $Q_* = Q(-D/\tau)$ and recalling the definition of $Q(\alpha)$. $\square$

Obviously, when demand $D = 0$, the pricing measure Q_* reduces to the minimal entropy martingale measure Q_e. We thus have the following result as a corollary.

Corollary 3.4.4. *Assuming zero demand for the derivatives, then the competitive prices $p(0, G)$ is given by*

$$p(0, G) = E^{Q(0)}[G], \tag{3.29}$$

where $Q(0)$ is the minimal entropy martingale measure with respect to the historical measure P.

3.4.1 Properties of Competition-Based Pricing

In what follows, we investigate some properties of the competition-based prices. First of all, we observe that the competition-based pricing measure Q_* defined in (3.27) is independent of the agent's initial capital endowment.

No Arbitrage. Obviously, the competition-based pricing measure $Q_* \in \mathbb{P}_e$, and therefore, we have

$$\inf_{Q\in\mathbb{P}_e} E^Q[B] \leq E^{Q_*}[B] \leq \sup_{Q\in\mathbb{P}_e} E^Q[B],$$

for any bounded claim $B = \alpha \cdot G$. This shows that the competition-based price consists with the no-arbitrage principle.

The Pricing Kernel. From Theorem 3.4.3, we deduce the following pricing kernel

$$\xi(D, \tau; G) := \frac{dQ_*}{dP} = \exp\left\{v(-D/\tau; G) - \frac{1}{\tau}\left(X^*(D) - D \cdot G\right)\right\}, \tag{3.30}$$

where $X^*(D) = \int_t^T \vartheta_u^* dS_u$ is the terminal wealth of the optimal strategy. In terms of this pricing kernel, the derivative prices is represented by

$$p_i = E^{Q_*}[G^i] = E[\xi(D)G^i]. \tag{3.31}$$

Scaling Invariance. We have the following scaling property for the pricing kernel

$$\xi(kD, k\tau; G) = \xi(D, \tau; G).$$

Hedging Strategy. We next consider the optimal hedging strategies for the intermediaries. From the construction of the competitive price, we see that the optimal derivative position of intermediary i is given by

$$q_i^* = -\frac{\tau_i}{\tau} D = -D \Bigg/ \sum_{j \in I_t} \frac{\gamma_i}{\gamma_j}.$$

Thus the hedging strategy of intermediary i can be achieved by solving optimization problem

$$\sup_{\vartheta \in \Theta} E\left[-\exp\left\{-\gamma_i \left(X_T^{\vartheta, x_i - q_i^* \cdot p_t} + q_i^* \cdot G\right)\right\}\right].$$

It is worth noting that the hedging strategy is unique in the sense that its terminal value $X^*(D)$ is unique for given market data (D, G).

3.5 The Demand Effects on Derivative Prices

In this section we investigate the price effects of demand. We first recall that, in terms of the pricing kernel $\xi(D; G)$, the competitive price p_i of the derivative G^i is given by $p_i = E[\xi(D)G^i]$ with ξ defined in (3.30).

Theorem 3.5.1. *The partial derivative of p_i with respect to the demand D_j, $(i, j = 1, \cdots, n)$ is given by*

$$\frac{\partial p_i}{\partial D_j} = Cov_{Q_*}\left(\tilde{G^i}, \tilde{G^j}\right), \tag{3.32}$$

where

$$\tilde{G^i} = G^i - E_{Q_*}[G^i] - \frac{\partial X^*(D)}{\partial D_i}$$

is the unhedgeable part of the claim G^i *and* $E_{Q_*}[\tilde{G^i}] = 0$.

Proof. We first observe that $E[\xi(D;G)] = 1$. Taking derivative with respect to D_i yields

$$0 = E\left[\frac{\partial \xi(D)}{\partial D_i}\right] = \frac{1}{\tau} E\left[\xi(D)\left(G^i - E[\xi(D)G^i] - \frac{\partial X^*(D)}{\partial D_i}\right)\right]. \tag{3.33}$$

It follows that

$$E_{Q_*}[\tilde{G^i}] = 0 \quad \text{and} \quad E_{Q_*}\left[\frac{\partial X^*(D)}{\partial D_i}\right] = 0. \tag{3.34}$$

We again differentiate (3.33) with respect to D_j, and obtain

$$0 = E_{Q_*}\left[\tilde{G^i}\tilde{G^j}\right] - \frac{1}{\tau} E_{Q_*}\left[G^i \tilde{G^j}\right] - E_{Q_*}\left[\frac{\partial^2 X^*(D)}{\partial D_i \partial D_j}\right]. \tag{3.35}$$

Combining (3.34) and (3.35) yields

$$\frac{1}{\tau} E_{Q_*}\left[G^i \tilde{G^j}\right] = E_{Q_*}\left[\tilde{G^i}\tilde{G^j}\right] = \mathrm{Cov}_{Q_*}\left(\tilde{G^i}, \tilde{G^j}\right).$$

Now, from the pricing formula (3.31), we deduce that

$$\begin{aligned}
\frac{\partial p_i}{\partial D_j} &= E\left[\frac{\partial \xi(D)}{\partial D_j} G^i\right] \\
&= \frac{1}{\tau} E\left[\xi(D) G^i \left(G^j - E[\xi(D)G^j] - \frac{\partial X^*(D)}{\partial D_j}\right)\right] \\
&= \frac{1}{\tau} E_{Q_*}\left[G^i \tilde{G^j}\right] \\
&= \mathrm{Cov}_{Q_*}\left(\tilde{G^i}, \tilde{G^j}\right),
\end{aligned}$$

as desired. ∎

The above result shows that the price of a derivative is increasing with the demand of another derivative in a rate proportional to the covariance, under the competitive pricing measure, between the unhedgeable parts of the two associated derivative payoffs. In particular, the derivative price is increasing with its own demand pressure. This is helpful to resolve the *option-pricing puzzles* because it is evident that the demands of index options and deep out-of-the-money (OTM) puts are very high comparing to other individual equity options; see among others, [15, 21, 22, 56].

3.6 Relations to Existing Pricing Concepts

The concept of competition-based pricing has a broad connection to the prevailing concepts of pricing in the literature. In this section we investigate some of them.

The Marginal Utility-Based Pricing. The marginal indifference price was first introduced by [32] based on the idea of "zero marginal rate of substitution".[9] Considering a representative agent economy and a given contingent claim, the fair price $\hat{p}$ is defined such that the agent can not increase his expected utility by diverting an infinitesimal amount of his capital into the claim. Formally, we quote the following definition from [32].

Definition 3.6.1. *Consider a single European claim B and a representative agent who wants to maximize the expected utility of terminal wealth:*

$$u(x, \delta; B) = \sup_{\vartheta \in \Theta} E\left[U(X_T^{\vartheta,x} + \delta B)\right],$$

where $U(x)$ is the agent's utility function satisfying the Inada condition. Then the marginal price $\hat{p}(B)$ of the claim B satisfies

$$\frac{\partial u}{\partial \delta}(x - \delta\hat{p}, \delta; B)\Big|_{\delta=0} = 0.$$

Under exponential utility $U(x) := -e^{-\gamma x}$, it has been shown that the fair price is given by

$$\hat{p} = E^{Q(0)}[B],$$

[9] There are many terminologies arising from this pricing concept such as "marginal (indifference) price", "(marginal) utility-based price", "Davis' fair price", "neutral price" as in [79, 80], and "shadow price" as in [51].

for a bounded claim B. This coincides with the competition-based price when we set the demand pressure equal zero and the aggregate risk tolerance parameter equal $1/\gamma$. In other words, the marginal price can be interpreted as a special case of competition-based price. It corresponds to the case of single agent, single derivative, and zero demand pressure. Therefore the concept of competition-based price generalizes the framework of marginal utility price.

The fair price can be also viewed as the limit of indifference price for vanishing risk-aversion (say, infinitesimal contract volume).

3.7 Examples in Markovian Diffusion Framework

In this section, we consider a continuous-time financial market consisting of a riskless bond B (money-market account) and a risky stock S, whose prices are given exogenously. The uncertainty of the world is described by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, on which is defined a standard two-dimensional Brownian motion $(W_t^1, W_t^\perp; t \geq 0)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the P-augmented filtration generated by $(W^1, W^\perp)$. The stock price is assumed to follow a diffusion process satisfying

$$dS_t = \mu(t, Y_t) S_t dt + \sigma(t, Y_t) S_t dW_t^1, \quad t \geq 0, \tag{3.36}$$

with $S_0 > 0$. The drift μ and volatility σ of the stock are driven by a stochastic factor Y, which is modelled as a correlated diffusion

$$dY_t = b(t, Y_t) dt + a(t, Y_t)(\rho dW_t^1 + \bar{\rho} dW_t^\perp), \quad t \geq 0, \tag{3.37}$$

with $\rho \in (-1, 1)$ being the correlation coefficient and $\bar{\rho} = \sqrt{1-\rho^2}$. The bond is assumed to mature at T and be tradable over the time horizon $[0, T]$, yielding constant interest rate r. Without loss of generality we take $r = 0$, which is equivalent to use the bond as numeraire. The results for $r > 0$ follow directly from the standard rescaling arguments and are not presented herein.

The market coefficients $\mu(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $a(\cdot,\cdot)$ are assumed to satisfy all the regularity conditions such that equations (3.36) and (3.37) have a unique strong solution satisfying $S_s > 0$ P-a.s. for a.e. $s \in [t, T]$. The *Sharpe ratio* process of the stock is defined by $\lambda_t = \lambda(t, S_t), t \geq 0$ with $\lambda(t, S) := \mu(t, S)/\sigma(t, S)$.

We denote by (π, x) a self-financing portfolio with initial capital x and π

being the amount invested in the stock. Then, direct calculation shows that the corresponding *wealth* process $X^{\pi,x}$ satisfies the following controlled diffusion equation

$$dX_s = \mu(s, Y_s)\pi_s ds + \sigma(s, Y_s)\pi_s dW_s^1, \quad t \le s \le T, \tag{3.38}$$

with $X_t = x \in \mathbb{R}$ [see 96]. The amount invested in the stock (π_s) is the only control variable that represents a trading strategy. A control process π is called admissible if it is $\mathcal{F}_s$-predictable and satisfies the integrability condition $E\int_0^T \sigma^2(s, Y_s)\pi_s^2 ds < +\infty$. We denote by $\mathcal{A}$ the set of all admissible trading strategies.

Besides the fundamental securities, there are also n $(n \ge 1)$ derivatives (contingent claims) available in the market, which are characterized in terms of their terminal payoff at maturity T. The derivative payoffs are given exogenously by n random variables $(G^1, \cdots, G^n)$ that are contingent only on Y. To avoid triviality, we assume that all these derivatives are non-redundant in the sense of (3.3).

In this setting, the intermediaries' optimization problem (3.5) can be written as

$$\max_{q_i \in \mathbb{R}^n} u(t, x_i - q_i \cdot p, q_i; \gamma_i), \tag{3.39}$$

where we have introduced the value function

$$u(t, x, y, q; \gamma) := \sup_{\pi \in \mathcal{A}} E\left[-\exp\left\{-\gamma\left(X_T^{\pi,x} + q \cdot G\right)\right\} \middle| \mathcal{F}_t\right], \tag{3.40}$$

with $X^{\pi,x}$ being the wealth dynamics defined in (3.38) and $\mathcal{A}$ the set of admissible trading strategies.

For fixed G and q, the valuation problem (3.40) evaluates the maximal utility of terminal wealth arising from the derivative payoff $q \cdot G$ and the terminal fund that can be achieved by trading optimally in the underlyings with initial capital x. Such valuation problem and its variations have been studied extensively in mathematical finance, in particular, in the context of indifference pricing. Stochastic control is one of the most popular approaches, which studies the problem through the associated Hamilton-Jacobi-Bellman (HJB) equations. Thanks to the convenience of exponential utility, we are able to construct the classical solutions for such fully nonlinear equations using separation of variables. It follows from a standard argument of viscosity solution that these solutions must coincide with the respective value function. In the sequel, we analyze the valuation problem (3.40) following the similar approach used in [111].

To this end, the associated Hamilton-Jacobi-Bellman (HJB) equation is given by

$$u_t + \max_{\pi} \left(\frac{1}{2}\sigma^2\pi^2 u_{xx} + \pi\left(\rho\sigma a u_{xy} + \mu u_x\right) \right) + \frac{1}{2}a^2 u_{yy} + b u_y = 0, \tag{3.41}$$

for $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, with $u(T, x, y) = -e^{-\gamma(x+q\cdot g(y))}$. Moreover, following similar arguments used in Theorems 4.1 and 4.2 of [46], we find that the value function u defined in (3.40) is the unique viscosity solution of (3.41) in the class of functions that are concave, increasing in x, and uniformly bounded in y for any fixed (t, x). We now proceed to solve the HJB equation (3.41).

Proposition 3.7.1. *The value function u can be further represented by*

$$u(t, x, y; q, \gamma) = -e^{-\gamma x} f(t, y; q, \gamma)^{\frac{1}{1-\rho^2}}, \tag{3.42}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ solves the linear equation

$$f_t + \frac{1}{2}a^2(t, y) f_{yy} + \left[b(t, y) - \rho\lambda a(t, y)\right] f_y = \frac{1}{2}(1 - \rho^2)\lambda^2(t, y) f, \tag{3.43}$$

with $f(T, y; q, \gamma) = \exp\{-\gamma\bar{\rho}^2 q \cdot g(y)\}$ for all $y \in \mathbb{R}$.

Proof. The proof follows along the similar arguments used in Theorem 2.2 in [129] and hence omitted. □

To prepare for the probabilistic representation of the value function u, we recall that under stochastic volatility model the density of an *equivalent local martingale measure* (ELMM) Q is given by

$$\frac{dQ}{dP} = \exp\left\{ -\int_0^T \lambda(s, Y_s) dW_s^1 - \int_0^T \varphi_s dW_s^{\perp} - \frac{1}{2}\int_0^T \left(\lambda^2(s, Y_s) + \varphi_s^2\right) ds \right\}, \tag{3.44}$$

where P is the historical measure, λ is the Sharpe ratio process of the stock, and φ is an adapted process satisfying $\int_0^T \varphi_s^2 ds < +\infty$ a.s. [12]. We assume that

$E[dQ/dP] = 1$ so that Q is a probability measure equivalent to P on $\mathcal{F}_T$. A sufficient condition to ensure this is the Novikov condition

$$E\left[\exp\left\{\frac{1}{2}\int_0^T \lambda^2(s, Y_s)ds\right\}\right] < +\infty. \tag{3.45}$$

We denote by $\mathcal{M}$ the set of all ELMM. It is worth noting that the set $\mathcal{M}$ is one-to-one correspondence to the set Φ of integrands φ. We write Q_φ to emphasis the dependence of Q on φ whenever needed. Under measure $Q \in \mathcal{M}$, the dynamics of S and Y are given by

$$\begin{aligned} dS_t =& \sigma(t, Y_t)S_t dW_t^{1,Q}, \\ dY_t =& (b(t, Y_t) - a(t, Y_t)(\rho\lambda(t, Y_t) + \bar{\rho}\varphi_t))\, dt \\ &+ a(t, Y_t)\left(\rho dW_t^{1,Q} + \bar{\rho} dW_t^{\perp,Q}\right), \end{aligned} \tag{3.46}$$

where $(W^{1,Q}, W^{\perp,Q})$ is a two-dimensional standard Brownian motion defined by

$$W_t^{1,Q} := W_t^1 + \int_0^t \lambda(s, Y_s)ds, \tag{3.47}$$

$$W_t^{\perp,Q} := W_t^{\perp} + \int_0^t \varphi_s ds. \tag{3.48}$$

Among all ELMM, the *minimal entropy martingale measure* (MEMM) is closely related to the exponential utility maximization [see 40]. We recall that the *relative entropy* $H(Q|P)$ of any probability measure Q with respect to P is defined as

$$H(Q|P) := \begin{cases} E\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)|\mathcal{F}_t\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P. It is worth noting that the relative entropy $H(Q|P)$ measures the distance between the two probability distributions Q and P, which is always nonnegative [see 75]. The entropy of an ELMM $Q \in \mathcal{M}$ is

$$H(Q_\varphi|P) = E_{Q_\varphi}\left[\frac{1}{2}\int_0^T \left(\lambda^2(s, Y_s) + \varphi_s^2\right) ds\right]. \tag{3.49}$$

Obviously, there exists an ELMM with finite entropy. Thus there is a unique martingale measure $Q_e \in \mathcal{M}$ minimizing the relative entropy $H(Q|P)$ over all $Q \in \mathcal{M}$, nemely

$$Q_e = \underset{Q \in \mathcal{M}}{\arg\min}\, H(Q|P). \tag{3.50}$$

See Theorem 2.2 and 2.3 of [54] and Proposition 3.2 and the proof of Theorem 4.3 of [64].

Another important martingale measure arising from our valuation problem is the so called *minimal martingale measure* (MMM) Q_0, whose density is given by

$$\frac{dQ_0}{dP} = \exp\left\{-\int_0^T \lambda(s, Y_s) dW_s^1 - \frac{1}{2}\int_0^T \lambda^2(s, Y_s) ds\right\}, \tag{3.51}$$

which corresponds to take $\varphi = 0$ in (3.44). This measure was originally introduced by [52]. Under measure Q_0 the discounted traded asset is a martingale while the law of the orthogonal martingale measure remains unchanged. For models with continuous price process, it has been shown that Q_0 is the martingale measure minimizing the reverse entropy $H(\mathbb{P}|Q)$ over all ELMM $Q \in \mathcal{M}$, i.e.

$$Q_0 = \underset{Q \in \mathcal{M}}{\arg\min}\, H(P|Q) := \underset{Q \in \mathcal{M}}{\arg\min}\, E_P\left(-\log\frac{dQ}{dP}\right). \tag{3.52}$$

We refer the reader to [125, 126] for more discussions. We are now ready to prove the following representation of the value function u.

Theorem 3.7.2. *The value function u is given by*

$$u(t, x; q; \gamma) = -\exp\left\{-\gamma(x + w(t, y; q; \gamma))\right\}, \tag{3.53}$$

where

$$\begin{aligned} w(t, y; q; \gamma) = &-\frac{1}{\gamma\bar{\rho}^2}\log E_{Q_e}^{t,y}\left[e^{-\gamma\bar{\rho}^2 q \cdot g(Y_T)}\right] \\ &-\frac{1}{\gamma\bar{\rho}^2}\log E_{Q_0}^{t,y}\left[\exp\left\{-\frac{1}{2}\bar{\rho}^2\int_t^T \lambda^2(s, Y_s) ds\right\}\right], \end{aligned} \tag{3.54}$$

with Q_e being the minimal entropy martingale measure defined by

$$\frac{dQ_e}{dP}\Big|_{\mathcal{F}_t} = \frac{\exp\left\{-\int_0^T \lambda(s,Y_s)dW_s^1 - \frac{1}{2}(1+\bar{\rho}^2)\int_0^T \lambda^2(s,Y_s)ds\right\}}{E^{t,y}\left[\exp\left\{-\int_0^T \lambda(s,Y_s)dW_s^1 - \frac{1}{2}(1+\bar{\rho}^2)\int_0^T \lambda^2(s,Y_s)ds\right\}\right]}. \tag{3.55}$$

Proof. By Girsanov's theorem, the dynamics of S and Y under MMM Q_0 are given by

$$dS_t = \sigma(t,Y_t)S_t d\tilde{W}_t^1, \tag{3.56}$$

$$dY_t = (b(t,Y_t) - \rho\lambda a(t,Y_t))dt + a(t,Y_t)(\rho d\tilde{W}_t^1 + \bar{\rho} dW_t^{\perp}), \tag{3.57}$$

where $\tilde{W}_t^1 = W_t^1 + \int_0^t \lambda(s,Y_s)ds$ is a Q_0-Brownian motion independent with $W_t^{\perp}$. Using Feynman-Kac formula, we find that the solution $f(t,y)$ to the equation (3.43) admits the probabilistic representation

$$f(t,y) = E_{Q_0}\left[\exp\left\{-\frac{1}{2}\bar{\rho}^2\int_t^T \lambda^2(s,Y_s)ds - \gamma\bar{\rho}^2 q\cdot g(Y_T)\right\}\Big| Y_t = y\right], \tag{3.58}$$

for $(t,y) \in [0,T]\times\mathbb{R}$.

We next define a measure Q_e through density

$$\frac{dQ_e}{dQ_0}\Big|_{\mathcal{F}_t} = \frac{\exp\left\{-\frac{1}{2}\bar{\rho}^2\int_0^T \lambda^2(s,Y_s)ds\right\}}{E_{Q_0}^{t,y}\left[\exp\left\{-\frac{1}{2}\bar{\rho}^2\int_0^T \lambda^2(s,Y_s)ds\right\}\right]}, \tag{3.59}$$

which is sometimes called Esscher change of measure. Combining (3.58) and (3.59), we find that

$$f(t,x;q;\gamma) = E_{Q_e}^{t,y}\left[e^{-\gamma\bar{\rho}^2 q\cdot g(Y_T)}\right] E_{Q_0}^{t,y}\left[\exp\left\{\frac{1}{2}\bar{\rho}^2\int_t^T \lambda^2(s,Y_s)ds\right\}\right].$$

Denote $w(t,y;q,\gamma) = -\frac{1}{\gamma\bar{\rho}^2}\log f(t,y)$, then direct calculation show that w is given by (3.54). Thus we conclude the representation (3.53) by recalling (3.42). The fact that Q_e is the minimal entropy martingale measure has been well established; see [54, 6, 64], and [108]. We finally combine (3.51) and (3.59) to derive the density (3.55). □

It is useful to observe that the value function $w(t,y;q,\gamma)$ is strictly concave in

q for all $(t, y) \in [0, T] \times \mathbb{R}$. To see this, we recall that the function $F(z_1, z_2) = z_1 z_2$ is convex on $\mathbb{R}^+ \times \mathbb{R}^+$ and the concavity follows by applying Jensen's inequality. Now we are ready to prove the following results.

Theorem 3.7.3. *The intermediaries' optimal investment problems (3.39) admit a unique solution (π_i, q_i), where the optimal derivative holdings q_i are characterized by the first order condition*

$$E_{Q_e}\left[e^{-\gamma_i(1-\rho^2)q_i\cdot g(Y_T)}\big(g(Y_T) - p_t\big)\Big|Y_t = y\right] = 0, \tag{3.60}$$

where Q_e is the minimal entropy martingale measure defined in (3.55).

Proof. The existence and uniqueness of π_i have been well established. For those of q_i, we observe from (3.39) and (3.53) that

$$\max_{q_i\in\mathbb{R}^n} u\left(t, x_i - q_i \cdot p, y; q_i, \gamma_i\right) = -e^{-\gamma_i x_i} \exp\left\{-\gamma_i \max_{q_i\in\mathbb{R}^n} \big(w(t, y; q_i, \gamma_i) - q_i \cdot p\big)\right\},$$

where w is given in (3.54). Thus, the existence and uniqueness of q_i follow directly from the concavity of w.

The first-order condition of u is given by

$$\begin{aligned} 0 &= \nabla_q u(t, x_i - q_i \cdot p, y; q_i, \gamma_i) \\ &= \gamma u\left[p - \nabla_q w(t, y; q_i, \gamma_i)\right], \end{aligned}$$

which yields that

$$p = \nabla w(t, y; q_i, \gamma_i) = \frac{E_{Q_e}\left[e^{-\gamma_i(1-\rho^2)q_i\cdot g(Y_T)}g(Y_T)\Big|Y_t = y\right]}{E_{Q_e}\left[e^{-\gamma_i(1-\rho^2)q_i\cdot g(Y_T)}\Big|Y_t = y\right]}, \tag{3.61}$$

and completes the proof. □

In the next theorem, we summarize the competition-based pricing formula and the associated pricing measure.

Theorem 3.7.4. *Let τ denote the aggregate risk tolerance parameter of the intermediaries in the economy, namely, $\tau := \sum_{i\in I_t} \tau_i = \sum_{i\in I_t} \frac{1}{\gamma_i}$, then the competition-based* *price $p(G,D)$ is given by*

$$p_t = E_{Q_*}\left[g(Y_T)|Y_t = y\right], \tag{3.62}$$

where the pricing measure Q_* with density given by

$$\frac{dQ_*}{dQ_e} = \frac{\exp\left\{\frac{\bar{\rho}^2}{\tau} D \cdot g(Y_T)\right\}}{E_{Q_e}\left[\exp\left\{\frac{\bar{\rho}^2}{\tau} D \cdot g(Y_T)\right\}\right]}, \tag{3.63}$$

is an Esscher transform of the minimal entropy martingale measure Q_e.

Proof. We first take the price vector p_t as given and denote by α_p the unique solution to the equation

$$E_{Q_e}[e^{-\bar{\rho}^2\alpha \cdot g(Y_T)}(g(Y_T) - p_t)|Y_t = y] = 0,$$

where Q_e is given by (3.55). Thus the first-order condition (3.60) shows that the optimal derivative positions q_i of the intermediary i satisfy $\gamma_i q_i = \alpha_p$, i.e.

$$q_i = \frac{\alpha_p}{\gamma_i} = \alpha_p \tau_i.$$

Combining the market clearing condition (3.6) yields that

$$\alpha_p = -\frac{D}{\sum_{i \in I_t} \tau_i} = -\frac{D}{\tau}.$$

To this end, the first-order condition (3.60) again shows that the equilibrium price p_t is given by

$$p_t = \frac{E_{Q_e}\left[e^{-\frac{\bar{\rho}^2}{\tau} D \cdot g(Y_T)} g(Y_T) \middle| Y_t = y\right]}{E_{Q_e}\left[e^{-\frac{\bar{\rho}^2}{\tau} D \cdot g(Y_T)} \middle| Y_t = y\right]}. \tag{3.64}$$

Therefore, we conclude by denoting recalling the definition of Q_*. $\square$

From the above result follows immediately that for $D = 0$, the pricing measure Q_* becomes identical to the minimal entropy martingale measure Q_e. In this case, the competition-based price reduces to the marginal utility price, namely,

$p_t = E_{Q_e}[g(Y_T)|Y_t]$.

Remark 3.7.5. *From Theorem 3.4.3, we see taht the pricing measure Q_* can be as well characterized as follows. Define the probability measure $P_* \sim P$ through the Esscher transform*

$$\frac{dP_*}{dP} = \exp\left\{\frac{1}{\tau}D \cdot G\right\} \Big/ E\left[\exp\left\{\frac{1}{\tau}D \cdot G\right\}\right], \tag{3.65}$$

where D is the aggregate demand of the derivatives G. Then the pricing measure Q_* coincides with the minimal entropy martingale measure with respect to the new prior P_*, namely,

$$Q_* = \underset{Q \in \mathcal{M}}{\arg\min}\, H(Q|P_*) := \underset{Q \in \mathcal{M}}{\arg\min}\, E_Q\left[\log \frac{dQ}{dP_*}\right]. \tag{3.66}$$

In the sequel, we follow Section 3.5 to derive the price effects of demand. In current setting, the pricing kernel is defined as

$$\xi(D; \tau, G) := \frac{dQ_e}{dP} \cdot \frac{dQ_*}{dQ_e}, \tag{3.67}$$

where $\frac{dQ_e}{dP}$ and $\frac{dQ_*}{dQ_e}$ are given by (3.55) and (3.63), respectively. In terms of pricing kernel $\xi(D)$, the competitive price of the derivative G^j is given by $p_j = E[\xi(D)G^j]$. We have the following result.

Theorem 3.7.6. *The partial derivative of p_j with respect to the demand D_k is given by*

$$\frac{\partial p_j}{\partial D_k} = \frac{\bar{\rho}^2}{\tau} Cov^{t,y}_{Q_*}\left(G^j, G^k\right) = \frac{1}{\tau} Cov^{t,y}_{Q_*}\left(\tilde{G^j}, \tilde{G^k}\right), \tag{3.68}$$

where $\tilde{G^j} = \bar{\rho} G^j$ is the unhedgeable part of the claim G^j.

Proof. We recall that under the pricing kernel $\xi(D)$, the competitive price

$$p_j = E\left[\xi(D)G^j \middle| \mathcal{F}_t\right] = E^{t,y}_{Q_e}\left[\frac{dQ_*}{dQ_e} G^j\right].$$

We then differentiate the above equation with respect to D_k to deduce

$$\begin{aligned}\frac{\partial p_j}{\partial D_k} &= E^{t,y}_{Q_e}\left[\frac{\partial}{\partial D_k}\left(\frac{dQ_*}{dQ_e}\right)G^j\right]\\ &= \frac{\bar{\rho}^2}{\tau}E^{t,y}_{Q_e}\left[\frac{dQ_*}{dQ_e}\left(G^k - E^{t,y}_{Q_e}\left[\frac{dQ_*}{dQ_e}G^k\right]\right)G^j\right]\\ &= \frac{\bar{\rho}^2}{\tau}E^{t,y}_{Q_*}\left[G^jG^k - G^jE^{t,y}_{Q_*}[G^k]\right]\\ &= \frac{\bar{\rho}^2}{\tau}\left(E^{t,y}_{Q_*}[G^jG^k] - E^{t,y}_{Q_*}[G^j]E^{t,y}_{Q_*}[G^k]\right)\\ &= \frac{\bar{\rho}^2}{\tau}\mathrm{Cov}^{t,y}_{Q_*}\left(G^j, G^k\right)\\ &= \frac{1}{\tau}\mathrm{Cov}^{t,y}_{Q_*}\left(\tilde{G}^j, \tilde{G}^k\right),\end{aligned}$$

as desired. □

3.8 Conclusions

We have developed a competitive equilibrium approach to price derivatives in an incomplete semimartingale market, which consists of arbitrary numbers of fundamental securities and non-redundant derivatives.

Based on Merton's functional perspective of financial intermediation, we model the competition among an arbitrary number of financial intermediaries in an incomplete semimartingale market and analyze the price effects of aggregate demand by their customers.

Our model shows that the price of a derivative is increasing with the demand of another derivative in a rate proportional to the covariance between the unhedgeable parts of the two associated derivative payoffs, calculated under the competitive pricing measure. In particular, the price of a derivative is increasing with its demand in a rate proportional to the variance of its unhedgeable part.

Part II

Indifference Approach

Chapter 4

Indifference Valuation for A Stream of Contingent Claims

4.1 Introduction

Indifference valuation is, by now, one of the popular derivative pricing methodologies in incomplete markets. The central idea is to produce the notion of value by replacing the traditional replication argument by the investment optimality. Generally speaking, the indifference price is the amount of money that makes a rational investor indifferent between the investment opportunities with and without holding the derivative. Mathematically, indifference prices are characterized by two expected utility maximization problems with constraints. This valuation concept was introduced originally by [72] in analyzing the impact of transaction costs on the price. Indifference prices have been extensively studied through both primal and dual forms of the underlying stochastic optimization problems.

The primal approach is based on stochastic control and dynamic programming technique. The underlying expected utility maximization problem with an additional liability have been studied by many authors. For example, [35] analyzed the associated Hamilton-Jacobi-Bellman equations. More results on hedging in incomplete models with intermediate consumption can be find in [44]. With a single lognormal stock dynamics, [33, 34, 68] and [137] studied the indifference pricing and hedging of a single European claim on nontraded assets and obtained explicit solutions. More recently, [111] derived a quasilinear PDE for the indifference price and obtained a nonlinear expectation pricing functional with the associated hedging strategy. [71] applied the approach to price real options. Indifference pricing is also studied in stochastic volatility model, see for instance, [129]. In parallel with continuous setting, [112] proposed an indifference valuation algorithm based on binomial model and provided a pricing model that values two types of risk.

The dual approach characterizes the optimal solutions in terms of certain measures and entropic criteria by using duality technique. The related utility maximization problems have been addressed in a semimartingale setting by [53] and [115] for general utility functions, and [40] and [78] for exponential utility. See also [82, 83, 84], and [121] for more dual treatment in utility maximization. Within Brownian setting, [118] resolved some technical difficulties and provided new proofs of the indifference valuation. In comparison with the primal approach, duality treatment losses intuitive results in paying for the increase of generality. In fact, little result about hedging strategies is provided except for existence. Neither explicit nor constructive pricing formula is obtained.

In this chapter, we first generalize the benchmark model of indifference price by [111] allowing for n European claims with distinct maturities. We provide the additional generalization of allowing the claim payoffs to depend on both tradable and nontraded assets, with levels being modelled by Itô diffusion processes. The goal herein is to introduce the concepts and notations of indifference pricing, as well as develop the fundamental tools and results in this basic setting for convenience of future reference.

We specify the two underlying stochastic optimization problems that characterize the indifference price. By using techniques from dynamic programming, we derive the Hamilton-Jacobi-Bellman equations for the value functions and a quasilinear PDE for the indifference price. Under appropriate market restrictions, we obtain explicit pricing formulae in a form of nonlinear expectation. The pricing measure turns out to be the one, among all equivalent martingale measures, that minimizes the relative entropy with respect to the historical one. Using the pricing PDE, we build up useful properties and robustness of the indifference price. We then analyze hedging and risk monitoring strategies with comparison to the Black and Scholes model. A payoff decomposition result is further developed through the associated feedback optimal policy, residual optimal wealth and residual risk.

However, the classical indifference approach concentrates mainly on a single contract. Neither investors' *ex-ante* risk exposure nor their *ex-post* investment opportunities are taken into account. In many cases, on the other hand, there are multiple options or opportunities available for investment. Evaluating each contract in isolation, investors not only may misestimate the value of the contract due to disregarding her preexistent risk exposure, but also may throw away the benefits of their private information about future investment opportunities. When evaluating the current contract, a *foresighted* investor should consider both her ex-ante portfolio holdings and ex-post investment opportunities. This motivates us to construct the the relative and foresighted indifference prices.

Following [109] and [131], the relative indifference price is defined as the amount of money that compensates exactly a utility maximizer, who holds ex ante a portfolio of options, to take the risk of holding an additional derivative. Similar to the development of classical indifference price, we derive a nonlinear PDE for the relative price. It turns out that the price explicitly depends on not only the investor's risk aversion but also her ex-ante portfolio holdings. The properties of the relative price and the risk monitoring strategy are further developed. Finally, we define and analyze the *foresighted indifference price.* This foresighted concept incorporates investor's private information into pricing and hedging, which is a generalization of the classical and relative indifference pricing.

4.2 Market and Investment Models

We consider a dynamic investment environment with a riskless bond $\mathbf{B}$ and two risky assets: a tradable stock S and a nontraded asset Y. For simplicity, we assume constant interest rate $r = 0$.[1] The stock price is modelled via an Itô diffusion

$$dS_s = \mu(s, S_s)S_s ds + \sigma(s, S_s)S_s dW_s^1, \quad t \leq s, \tag{4.1}$$

with $S_t = S > 0$. The level of the nontraded asset Y is given by a diffusion process

$$dY_s = b(s, Y_s)ds + a(s, Y_s)dW_s, \quad t \leq s, \tag{4.2}$$

[1] The results for $r > 0$ follow directly from the rescaling arguments.

with $Y_t = y \in \mathbb{R}$, where the coefficients b and a satisfy the Lipschitz and growth conditions such that equation (4.2) admits a unique strong solution.

In the above state equations, the processes W^1 and W are standard Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with usual conditions. They are correlated with coefficient $\rho \in (-1, 1)$, so that W_t can be written as $W_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^{\perp}$, with $(W^1, W^{\perp})$ being a two-dimensional Brownian motion on the probability space. The assumptions on the coefficients μ and σ are such that the equation (4.1) has a unique strong solution satisfying $S_s > 0$ a.s. for $s \in [t, T]$. We denote the *Sharpe ratio* process of the stock by $\lambda_s = \lambda(s, S_s), t \leq s$, where $\lambda(t, S) := \mu(t, S)/\sigma(t, S)$.

We next introduce a portfolio of n ($n \geq 1$) European claims written on both of the risky assets S and Y. Each claim $C_{T_i}, (i = 1, \cdots, n)$ generates a payoff $c_i(S_{T_i}, Y_{T_i})$ at its maturity $T_i \in [0, T]$, where the function c_i is bounded. We denote such a portfolio by

$$\mathcal{C}_n(S, Y) := \{c_1(S_{T_1}, Y_{T_1}), \cdots, c_n(S_{T_n}, Y_{T_n})\}. \tag{4.3}$$

Note that no restriction is imposed on the order of the claim's maturities.

Consider an investor with risk preferences modelled via an exponential utility function

$$u(x) = -e^{-\gamma x}, \quad x \in \mathbb{R}, \tag{4.4}$$

with positive risk aversion parameter γ, who aims to maximize the expected utility of her terminal wealth at time T. The investor can trade in the financial market in time horizon $[t, T]$, using a dynamic self-financing strategy for manipulating the balances of the bond and stock accounts. However, no exogenous fund except for the claim payoffs is available, nor is intermediate consumption allowed.

Starting at time t with initial wealth $x \in \mathbb{R}$, the investor holds Δ_s shares of stock at time $s \in [t, T]$ with spot value of $\pi_s = \Delta_s S_s$, and deposits the remainder of her wealth into the bond account. Thus, trading plainly in the financial market, the investor's total wealth (X_s^0) satisfies the following controlled diffusion

$$dX_s = \mu(s, S_s)\pi_s ds + \sigma(s, S_s)\pi_s dW_s^1, \tag{4.5}$$

for $s \in [t,T]$, with initial condition $X_t^0 = x \in \mathbb{R}$ (see Merton, 1969). Note that the amount invested in the stock (π_s) is used as control policy, and the set of admissible policies $\mathcal{A}_{[t,T]}$ is defined by $\mathcal{A}_{[t,T]} := \{\pi : \pi_s \text{ is } \mathcal{F}_s\text{-predictive s.t. } E\int_t^T \sigma^2(s,S_s)\pi_s^2 ds < \infty\}$.

As is mentioned previously, the utility-indifference methodology is based on the comparison between two utility maximization problems with and without the presence of the claims. The first problem is the classical Merton model of optimal investment, where the investor seeks to maximize the expected terminal utility without holding the claims. Formally, it is defined by

$$M(t,S,x) := \sup_{\pi\in\mathcal{A}_{[t,T]}} E\left[u(X_T^0)\middle|S_t = S, X_t^0 = x\right], \tag{4.6}$$

for $(t,S,x) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$, with terminal condition $M(T,S,x) = u(x)$, where (X_s^0) is the wealth process satisfying (4.5).

The second problem is a utility maximization problem involving the portfolio $\mathcal{C}_n(S,Y)$. Suppose that the portfolio is introduced at time $t < \min_{1\le i\le n} T_i$. Although the investor can trade dynamically between the bond and the stock, no trading of the claims nor of the asset Y is allowed in the time horizon $[t,T]$. The corresponding stochastic optimization for holding the portfolio $\mathcal{C}_n$ is defined by

$$V^{\mathcal{C}_n}(t,S,x,y) := \sup_{\pi\in\mathcal{A}_{[t,T]}} E\left[u(X_T)\middle|S_t = S, X_t = x, Y_t = y\right], \tag{4.7}$$

for $(t,S,x,y) \in \mathcal{D}_T$, where (X_s) is the wealth process given by

$$X_s = x + \sum_{i=1}^n c_i(S_{T_i}, Y_{T_i})\mathbf{1}_{\{t<T_i\le s\}} + \int_t^s \mu(u,S_u)\pi_u du + \int_t^s \sigma(u,S_u)\pi_u dW_u^1, \tag{4.8}$$

for $s \in [t,T]$, where x stands for the investor's net wealth at time t. Note that the state domain $\mathcal{D}_T = \{(t,S,x,y) : t \in [0,T], S \in \mathbb{R}^+, x \in \mathbb{R}, y \in \mathbb{R}\}$. In the above wealth, we observe a jump at each claim maturity T_i whenever the claim payoff $c_i(S_{T_i}, Y_{T_i})$ is credited.

Next, we derive the value functions M and $V^{\mathcal{C}_n}$. For ease of presentation, we suppress the arguments of the market coefficients and introduce the differential operators

$$\mathcal{L}^{(S)} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \mu S \frac{\partial}{\partial S}, \tag{4.9}$$

$$\mathcal{L}^{(S,y)} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma a S \frac{\partial^2}{\partial S \partial y} + \frac{1}{2}a^2 \frac{\partial^2}{\partial y^2} + \mu S \frac{\partial}{\partial S} + b\frac{\partial}{\partial y}, \tag{4.10}$$

$$\tilde{\mathcal{L}}^{(S,y)} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma a S \frac{\partial^2}{\partial S \partial y} + \frac{1}{2}a^2 \frac{\partial^2}{\partial y^2} + (b - \rho\lambda a)\frac{\partial}{\partial y}. \tag{4.11}$$

It is straightforward to derive the Hamilton-Jacobi-Bellman equation for the Merton problem using standard techniques from dynamic programming and stochastic calculus. It turns out that the HJB equation can be linearized and solved by Feynman-Kac connection. We outline the main results as follows (a detail proof can be found in Appendix A.2).

Proposition 4.2.1. *The value function M defined in (4.6) solves the HJB equation*

$$M_t + \max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 M_{xx} + \pi\left(\sigma^2 S M_{Sx} + \mu M_x\right)\right) + \mathcal{L}^{(S)}M = 0, \tag{4.12}$$

for $(t, S, x) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}$, with terminal condition $M(T, S, x) = -e^{-\gamma x}$. It can be further represented by

$$M(t, S, x) = -\exp\{-\gamma(x + m(t, S))\}, \tag{4.13}$$

with

$$m(t, S) = \frac{1}{\gamma} E_{\mathbb{Q}}\left[\frac{1}{2}\int_t^T \lambda_s^2 ds \middle| S_t = S\right], \tag{4.14}$$

where $\lambda_s = \lambda(s, S_s)$ denotes the Sharpe ratio process of the stock, and the martingale measure $\mathbb{Q}$ is defined by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\int_0^T \lambda_s dW_s^1 - \frac{1}{2}\int_0^T \lambda_s^2 ds\right\}. \tag{4.15}$$

Moreover, the optimal trading policy $\Pi_s^{0,*}$ is given in the feedback form $\Pi_s^{0,*} =$

$\pi^{0,*}(s, S_s, X_s^{0,*})$ *for* $s \in [t, T]$*, where the feedback function* $\pi^{0,*}$ *is defined by*

$$\pi^{0,*}(t, S, x) = \frac{1}{\gamma}\frac{\lambda(t, S)}{\sigma(t, S)} - S m_S(t, S), \tag{4.16}$$

for $(t, S, x) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}$*. The optimal wealth process is then given by*

$$dX_s^{0,*} = \mu(s, S_s)\Pi_s^{0,*} ds + \sigma(s, S_s)\Pi_s^{0,*} dW_s^1, \tag{4.17}$$

for $s \in [t, T]$*, with initial condition* $X_t^{0,*} = x \in \mathbb{R}$*.*

It is worth noting that the measure $\mathbb{Q}$ is known as the minimal entropy measure, which is the martingale measure minimizing the entropy relative to the historical measure $\mathbb{P}$, i.e.

$$\mathcal{H}(\mathbb{Q}|\mathbb{P}) = \min_{Q \in \mathcal{M}} \mathcal{H}(Q|\mathbb{P}) := \min_{Q \in \mathcal{M}} E_{\mathbb{P}}\left(\frac{Q}{\mathbb{P}} \ln \frac{Q}{\mathbb{P}}\right), \tag{4.18}$$

with $\mathcal{M}$ being the set of all martingale measures. We refer the reader to Frittelli (2000a), Rouge and El Karoui (2000) and Delbaen et al. (2002) for further discussions. Clearly, when the market converge to the complete model, it becomes the unique risk neutral measure.

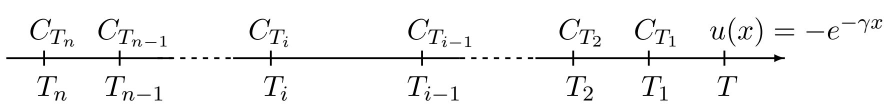


Figure 4.1: The payoff structure of a stream of claims

In what follows, we continue to construct the value functions $V^{\mathcal{C}_n}$. To facilitate the presentation, we assume for a moment that $T_i < T_{i-1}$ for $i = 1, \cdots, n$, as shown in Figure 4.1. Clearly, for $t \in (T_i, T_{i-1})$, the investor is faced with the optimal investment problem with portfolio $\mathcal{C}_{i-1} = \{c_1, \cdots, c_{i-1}\}$. The corresponding value function is given by $V^{\mathcal{C}_{i-1}}(t, S, x, y)$. At time T_i, for any wealth state X_{T_i} (determined by the control policy π up to T_i), the investor would enjoy the value of $V^{\mathcal{C}_{i-1}}(T_i, S_{T_i}, X_{T_i} + c_i(S_{T_i}, Y_{T_i}))$ since the derivative payoff $c_i(S_{T_i}, Y_{T_i})$ is credited at T_i. Thus for $t \in (T_{i+1}, T_i)$ and a given control policy $\pi \in \mathcal{A}_{[t,T_i]}$ up to time T_i, the buyer's expected utility payoff is given by[2]

$$J^\pi(t,S,x,y) := E\left[V^{\mathcal{C}_{i-1}}(T_i, S_{T_i}, X_{T_i} + c_i(S_{T_i}, Y_{T_i}))\middle| S_t = S, X_t = x, Y_t = y\right].$$

According to the Dynamic Programming Principle (DPP), the buyer's value function for $t \in (T_{i+1}, T_i)$ satisfies

$$\begin{aligned} V^{\mathcal{C}_i}(t,S,x,y) &= \sup_{\pi \in \mathcal{A}_{[t,T_i]}} J^\pi(t,S,x,y) \\ &= \sup_{\pi \in \mathcal{A}_{[t,T_i]}} E_t^{S,x,y}\left[V^{\mathcal{C}_{i-1}}(T_i, S_{T_i}, X_{T_i} + c_i(S_{T_i}, Y_{T_i}))\right], \end{aligned} \tag{4.19}$$

for $t \in (T_{i+1}, T_i]$, where the wealth process $X_s, s \in [t,T]$, is given by (4.8).

By using dynamic programming techniques and stochastic calculus, we derive the following HJB equation for $V^{\mathcal{C}_i}(t,S,x,y)$ (see Appendix A.2 for a detail derivation).

$$V_t + \max_\pi \left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \pi\left(\sigma^2 S V_{Sx} + \rho\sigma a V_{xy} + \mu V_x\right)\right) + \mathcal{L}^{(S,y)}V = 0, \tag{4.20}$$

for $t \in (T_{i+1}, T_i]$, with terminal condition

$$V^{\mathcal{C}_i}(T_i,S,x,y) = V^{\mathcal{C}_{i-1}}(T_i, S, x + c_i(S,y), y), \tag{4.21}$$

where operator $\mathcal{L}^{(S,y)}$ is defined in (4.10). Note that we've dropped the superscript $\mathcal{C}_i$ for simplicity and hereafter we'll do so whenever writing an equation for $V^{\mathcal{C}_i}$.

By plugging in the maximal control policy, substituting the ansatz form $V(t,S,x,y) = -\exp\{-\gamma(x + \phi(t,S,y))\}$, and applying induction to the recursive relation (4.19), we derive the quasilinear equation for ϕ and the optimal control policy. We further use the regularity properties of the value function V and classical verification results (see for example Theorem IV.3.1 in Fleming and Soner, 1993) to verify the optimality of our solution. The results are summarized as follows (see Appendix A.2 for a proof).

Theorem 4.2.2. *The value function $V^{\mathcal{C}_n}$ defined in (4.7) can be further represented*

[2] The value of the portfolio at time T_i after collecting the derivative payoff is given by $J^\pi(T_i, S_{T_i}, X_{T_i}, Y_{T_i}) = V^{\mathcal{C}_{i-1}}(T_i, S_{T_i}, X_{T_i} + c_i(S_{T_i}, Y_{T_i}))$ for a given $\pi \in \mathcal{A}_{[t,T_i]}$.

by

$$V^{\mathcal{C}_n}(t,S,x,y) = -\exp\{-\gamma(x+\phi(t,S,y))\}, \tag{4.22}$$

where the function $\phi : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *solves the quasilinear equation*

$$\phi_t + \tilde{\mathcal{L}}^{(S,y)}\phi - \frac{1}{2}\gamma(1-\rho^2)a^2\phi_y^2 + \frac{1}{2\gamma}\lambda^2 = 0, \tag{4.23}$$

with terminal condition $\phi(T^+,S,y) = 0$ *and pasting conditions* $\phi(T_i,S,y) = \phi(T_i^+,S,y) + c_i(S,y)$*, where* $\tilde{\mathcal{L}}^{(S,y)}$ *is given in (4.11).*

Moreover, the optimal trading policy Π_s^* *is given in the feedback form* $\Pi_s^* = \pi^*(s,S_s,X_s^*,Y_s)$ *for* $s \in [t,T]$*, where the feedback function* π^* *is defined by*

$$\pi^*(t,S,x,y) = \frac{1}{\gamma}\frac{\lambda(t,S)}{\sigma(t,S)} - S\phi_S(t,S,y) - \rho\frac{a(t,S)}{\sigma(t,S)}\phi_y(t,S,y), \tag{4.24}$$

for $(t,S,x,y) \in \mathcal{D}_T$*.*

The terminal condition of equation (4.23) can be written as $\phi(T,S,y) = c_0(S,y)$, where c_0 corresponds to the claim maturing at T. It can be zero if no claim in the original portfolio matured at T.

We conclude this section by providing an explicit formula for the value function $V^{\mathcal{C}_n}$ under additional assumptions on the market models. The following result is a natural extension of Theorem 3 in Musiela and Zariphopoulou (2004a). The proof can be found in Appendix A.2.

Proposition 4.2.3. *Assume that the stock dynamics (4.1) is a homogeneous SDE, namely,* $\mu(t,S) = \mu(t)$ *and* $\sigma(t,S) = \sigma(t)$*, and that all the derivative payoffs* $c_i(S_{T_i},Y_{T_i}) = c_i(Y_{T_i})$ *depend only on the nontraded asset* Y*, which is an Itô diffusion given by (4.2). Then, the buyer's value function* $V^{\mathcal{C}_n}$ *for the portfolio* $\mathcal{C}_n$ *can be represented as*

$$V^{\mathcal{C}_n}(t,x,y) = -\exp\left\{-\gamma\left(x + \mathcal{E}_{\mathbb{Q}}^{(t,T)}(\mathcal{C}_n)(y)\right) - \frac{1}{2}\int_t^T \lambda^2(s)ds\right\}, \tag{4.25}$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$*, where* $\mathcal{E}_{\mathbb{Q}}^{(t,T)}$ *is a nonlinear semi-group operator given* *by*

$$\mathcal{E}_{\mathbb{Q}}^{(t,T)}(\mathcal{C}_n)(y) = -\frac{1}{\gamma(1-\rho^2)} \ln E_{\mathbb{Q}}^{t,y}\left[\exp\left\{-\gamma(1-\rho^2)\sum_{i=1}^{n} c_i(Y_{T_i})\mathbf{1}_{\{t\le T_i\}}\right\}\right], \quad (4.26)$$

with martingale measure $\mathbb{Q}$ *defined by (4.15)*

4.3 The Classical Indifference Price

In this section, we construct the indifference buying price for a portfolio of claims. According to the classical notion of indifference pricing, Musiela and Zariphopoulou (2004a) defined the indifference selling price for a European claim written on a nontraded asset. The buying price for multiple claims on both tradable and nontraded assets is defined analogically as follows.

Definition 4.3.1. *The buyer's indifference price of the portfolio* $\mathcal{C}_n$ *specified in (4.3) is defined as the amount of money* $\nu(\mathcal{C}_n;\gamma)$ *such that the utility maximizer is indifferent between buying the portfolio* $\mathcal{C}_n$ *at time* $t \le \min_{1\le i\le n} T_i$ *at the price of* ν *and disregarding such a trading opportunity. That is, the indifference price* ν *must satisfy*

$$M(t,S,x) = V^{\mathcal{C}_n}(t,S,x-\nu(\mathcal{C}_n;\gamma),y), \quad (4.27)$$

for $(t,S,x,y) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$*, where the value functions* M *and* $V^{\mathcal{C}_n}$ *are defined in (4.6) and (4.7), respectively.*

Now we are ready to derive the indifference price. The following result follows directly from the value function representations (4.13), (4.22) and the indifference condition (4.27).

Theorem 4.3.2. *The indifference price* $\nu(\mathcal{C}_n;\gamma)$ *defined in (4.27) is given by* $\nu(\mathcal{C}_n;\gamma) = h(t,S,y)$*, where the function* $h:[0,T]\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ *solves the quasilinear equation*

$$h_t + \tilde{\mathcal{L}}^{(S,y)}h - \frac{1}{2}\gamma(1-\rho^2)a^2h_y^2 = -\sum_{i=1}^{n} c_i(S,y)\cdot\delta(t-T_i), \quad (4.28)$$

with terminal condition $h(T^+, S, y) = 0$, where $\tilde{\mathcal{L}}^{(S,y)}$ is given in (4.11) and δ is the Dirac function.

In the above equation, we introduce the Dirac δ-function to simplify the notation. It formally represents the pasting condition $h(T_i, S, y) = h(T_i^+, S, y) + c_i(S, y)$ at time T_i.

In general, the quasilinear equation (4.28) cannot be linearized owing to the combination of the nonlinearity and the high dimensionality. If the claim payoffs depend only on the nontraded asset, i.e. $c_i(S, y) = c_i(y)$, and the stock dynamics (4.1) is homogeneous, then the equation becomes

$$h_t + \frac{1}{2}a^2 h_{yy} + (b - \rho\lambda a)h_y - \frac{1}{2}\gamma(1-\rho^2)a^2 h_y^2 = -\sum_{i=1}^{n} c_i(y) \cdot \delta(t - T_i), \quad (4.29)$$

with terminal condition $h(T^+, y) = 0$. This one-dimensional equation can be linearized using a logarithmic transformation $h = \frac{1}{\gamma(1-\rho^2)} \ln v$. By the Feynman-Kac connection, the solution can be further represented by

$$h(t, y) = -\frac{1}{\gamma(1-\rho^2)} \ln E_{\mathbb{Q}} \left[\exp\left\{ -\gamma(1-\rho^2) \sum_{i=1}^{n} c_i(Y_{T_i}) \mathbf{1}_{\{t \le T_i\}} \right\} \middle| Y_t = y \right], \quad (4.30)$$

for $(t, y) \in [0, T] \times \mathbb{R}$.

On the other hand, if the claim payoff depends *only* on the tradable asset, namely $c_i(S, y) = c_i(S)$, then it reduces to the complete market setting. The pricing function turns out to solve the Black-Scholes PDE

$$h_t + \frac{1}{2}\sigma^2 S^2 h_{SS} = -\sum_{i=1}^{n} c_i(S) \cdot \delta(t - T_i), \quad (4.31)$$

with terminal condition $h(T^+, S) = 0$. Note that the interest rate is assumed to be zero.

Another notable feature of the equation (4.28) is that the nonlinear term is of quadratic form. It is also the only term involving the risk-aversion coefficient γ. By using the comparison theorem, one can show that the indifference price as a function of γ is non-decreasing. This result is very intuitive because the more risk-averse you are, the less you are willing to pay for a claim with intrinsic risk.

Thus the indifference price is bounded by

$$\nu(\mathcal{C}_n;\gamma) \leq \nu(\mathcal{C}_n;0), \tag{4.32}$$

where $\nu(\mathcal{C}_n;0)$ is the indifference price as $\gamma \to 0$.

By passing the limit as $\gamma \to 0$, it turns out that the risk neutral indifference price $\nu(\mathcal{C}_n;0) = h^0(t,S,y)$ solves the linear equation

$$h_t + \tilde{\mathcal{L}}^{(S,y)}h = -\sum_{i=1}^{n} c_i(S,y)\cdot\delta(t-T_i), \tag{4.33}$$

with terminal condition $h(T^+,S,y)=0$. This follows from the robustness of viscosity solutions for equation (4.28) and the uniqueness of solution to the linear equation (4.33).

Using Feynman-Kac connection and Girsanov theorem, we easily deduce the probabilistic representation of h^0:

$$h^0(t,S,y) = E_{\mathbb{Q}}\left[\sum_{i=1}^{n} c_i(S,y)\mathbf{1}_{\{t\leq T_i\}}\middle| S_t = S, Y_t = y\right], \tag{4.34}$$

with minimal entropy martingale measure specified in (4.15). In other words, as the buyer becomes risk neutral, the indifference price converges to the expectation of the payoff under the minimal entropy measure. This asymptotic behavior has been well studied under dual arguments by Rouge and El Karoui (2000), Delbaen et al (2002) and Becherer (2003).

The PDE of the indifference pricing function (4.28) can be further characterized by its absence of the drift in S and the adjustment of the drift in y. The absence of the drift in S is by no mean an accidental observation. Since the stock S is tradable, the ingredient of the derivative risk generated by the stock S can be fully hedged through trading in S. So by no-arbitrage argument the drift in S should be exactly the interest rate, which is zero by assumption. This observation is well known in the Black-Scholes framework. On the other hand, the adjustment of the drift in y reflects the correlation of S and Y. Although the asset Y is nontraded, the investor can hedge partially the risk component associated with Y by trading in S that is correlated to Y. As a result, the market price of risk related to Y is reduced by the amount of $\rho\lambda$, which leads to the drift term $\left(\frac{b}{a} - \rho\lambda\right) a h_y$. In fact, as $\rho \to 1$, the market becomes complete because of the perfect correlation between S and Y. So the market prices of the two risky assets must be consistent with each other, namely $\lambda = b/a$, to exclude arbitrage opportunity, and therefore the drift in y would vanish as well.

4.4 Properties of The Pricing Functional

In this section, we investigate some properties of the indifference pricing functional. For simplicity, we concentrate on a claim with the payoff $C = c(S_\tau, S_\tau)$ realized at $\tau \in [t, T]$. We first observe from the equation (4.28) that the pricing function h does not depend on x, i.e. the indifference price is

- *Independent of initial wealth*: $\nu(C, \gamma)$ does not depend on x.

With some manipulation on the quasilinear PDE (4.28), we can verify the following property.

- *Volume-scaling*: $\nu(\alpha C; \gamma) = \alpha \nu(C; \alpha\gamma)$ for $\alpha \in [0, 1]$.

Nonlinearity. It is clear that the indifference pricing functional $\nu(C; \gamma)$ is nonlinear in C. This is a directly result of the nonlinearity of the pricing equation. In particular, from the pricing function (4.30), we clearly have

$$\nu(C_1; \gamma) + \nu(C_2; \gamma) - \nu(C_1 + C_2; \gamma) \propto Cov_{\mathbb{Q}}\left(e^{-\gamma(1-\rho^2)c_1(Y_\tau)}, e^{-\gamma(1-\rho^2)c_2(Y_\tau)} \middle| Y_t = y\right) \neq 0,$$

unless the two claims C_1 and C_2 are uncorrelated under the condition $Y_t = y$. This leads to

- *Non-additivity*: $\nu(C_1 + C_2; \gamma) \neq \nu(C_1; \gamma) + \nu(C_2; \gamma)$ in general.

Although it is nonlinear in general, the pricing functional ν admits linearity in some special cases. For example, if S and Y are independent under $\mathbb{P}$, i.e. $\rho = 0$, then it follows that

$$\nu(c_1(S_\tau) + c_2(Y_\tau); \gamma) = \nu(c_1(S_\tau); \gamma) + \nu(c_2(Y_\tau); \gamma) = E_{\mathbb{Q}}[c_1(S_\tau)] + \nu(c_2(Y_\tau); \gamma).$$

Similarly, if Y is functionally dependent on S, then $\rho = 1$ and it turns out that in this case

$$\nu(c_1(S_\tau) + c_2(Y_\tau); \gamma) = E_{\mathbb{Q}}[c_1(S_\tau)] + E_{\mathbb{Q}}[c_2(Y_\tau)].$$

Translation Invariance. Moreover, directly from the PDE (4.28), it is easy to verify that the pricing functional ν is translation invariant with respect to monetary constant risks.

- *Translation invariance*: $\nu(C + k; \gamma) = \nu(C; \gamma) + k$ for constant k.

This turns out to be a key property of risk measures (See Artzner et al. 1999, and Föllmer and Schied 2002a). In fact, the following stronger property is achieved in indifference price.

- *Translation invariance w.r.t. hedgeable risk*: If $c(S_\tau, Y_\tau) = c_1(S_\tau, Y_\tau) + c_2(S_\tau)$, then $\nu(C; \gamma) = \nu(C_1; \gamma) + E_{\mathbb{Q}}[c_2(S_\tau)]$.

To see this, let's recall that $\nu(C_1; \gamma) = h^1(t, S, y)$ solves the PDE (4.28) with terminal condition $h^1(\tau, S, y) = c_1(S, y)$ while $E_{\mathbb{Q}}[c_2(S_\tau)] = h^2(t, S)$ satisfies linear equation (4.31) with terminal data $h^2(\tau, S) = c_2(S)$. Thus simple calculation can verify that $h(t, S, y) = h^1(t, S, y) + h^2(t, S)$ solves the PDE (4.28) with terminal condition $h(\tau, S, y) = c_1(S, y) + c_2(S)$. Thus the property follows from the uniqueness of the PDE (4.28).

Robustness and Regularity. We next consider an asymptotic complete market setting and demonstrate the robustness of the indifference price ν under the proper assumptions. By passing the limit as $\rho \to 1$ in equation (4.28), and imposing the additional condition $\lambda = \frac{b}{a}$ to exclude static arbitrage opportunity, we obtain the following linear equation in the limit:

$$h_t + \frac{1}{2}\left(\sigma S \frac{\partial}{\partial S} + a \frac{\partial}{\partial y}\right)\left(\sigma S \frac{\partial}{\partial S} + a \frac{\partial}{\partial y}\right) h = 0, \tag{4.35}$$

which recovers the Black-Scholes PDE. This follows from the robustness of viscosity solutions for equation (4.28) (See, for example, Proposition 4.1 in Lions, 1983) and the uniqueness of the above linear equation. This leads to

- *Robustness.* If the tradable asset S becomes the perfect proxy of the nontraded asset Y, namely, $\rho = 1$ and the *sharpe ratios* $\frac{b}{a} \equiv \frac{\mu}{\sigma}$, then the indifference price ν reduces to the Black-Scholes price.

In addition, as shown in the analysis of the last section, we have

- *Regularity with respect to γ.*

$$\lim_{\gamma \to 0^+} \nu(g;\gamma) = h^0(t,S,y) = E_{\mathbb{Q}}\left[c(S_\tau, Y_\tau)\middle| S_t = S, Y_t = y\right].$$

Monotonicity and Concavity. We conclude this section by listing the following properties of the indifference price.

- *Increasing in C*: $\nu(C_1;\gamma) \le \nu(C_2;\gamma)$ if $C_1 \le C_2$.
- *Decreasing in γ*: $\nu(C;\gamma_1) \ge \nu(C;\gamma_2)$ if $\gamma_1 \le \gamma_2$.
- *Superhomogenous in C*: $\nu(\alpha C;\gamma) \ge \alpha\nu(C;\gamma)$ for $\alpha \in [0,1]$;
  $\nu(\alpha C;\gamma) \le \alpha\nu(C;\gamma)$ for $\alpha \ge 1$.
- *Concavity*: $\nu(\alpha C_1 + (1-\alpha)C_2;\gamma) \ge \alpha\nu(C_1;\gamma) + (1-\alpha)\nu(C_2;\gamma)$ for $\alpha \in [0,1]$.

These properties are well established in the duality literature. See, for example, Rouge and El Karoui (2000), Delbaen et al (2002) and Becherer (2003).

4.5 Hedging Strategy and Payoff Decomposition

In this section, we construct the optimal hedging strategy and analyze the payoff decomposition with comparison to the Black-Scholes framework.

In complete markets, the derivative payoff can be decomposed into the Black-Scholes price and the proceeds from trading in the financial market. In incomplete markets, however, not all risk can be hedged by trading. Consequently, the derivative payoff should contain an additional component that corresponds to the unhedgeable risk. This residual risk turns out to be accumulated over time. It converges to 0 as the market becomes complete (say, $\rho \to 1$).

To construct the optimal hedging strategy, let's recall the fact the $\phi(t,S,y) = m(t,S) + h(t,S,y)$ for $t \in [0,T]$, which is implied by the indifference condition. It follows from (4.24) that the optimal control process $\Pi_s^*, s \in [t, T]$, is given by

$$\Pi_s^* = \frac{1}{\gamma} \frac{\lambda(s, S_s)}{\sigma(s, S_s)} - S_s m_S(s, S_s) - S_s h_S(s, S_s, Y_s) - \rho \frac{a(s, S_s)}{\sigma(s, S_s)} h_y(s, S_s, Y_s), \quad (4.36)$$

with its optimality following from the classical verification results (see for example Theorem IV.3.1 in Fleming and Soner, 1993). Recalling (4.8), we obtain the optimal wealth process

$$X_s^* = x - h(t, S, y) + \sum_{i=1}^{n} c_i(S_{T_i}, Y_{T_i}) \mathbf{1}_{\{t < T_i \le s\}} + \int_t^s \mu \Pi_u^* du + \int_t^s \sigma \Pi_u^* dW_u^1, \quad (4.37)$$

for $s \in [t, T]$, where μ and σ depend on (u, S_u). Respectively, as shown in Proposition 4.2.1, the optimal wealth process corresponding to the Merton problem is given by

$$X_s^{0,*} = x + \int_t^s \mu \Pi_u^{0,*} du + \int_t^s \sigma(u, S_u) \Pi_u^{0,*} dW_u^1, \quad (4.38)$$

for $s \in [t, T]$, where the optimal control process $\Pi_s^{0,*} = \pi^{0,*}(s, S_s)$ with feedback function given by (4.16).

To this end, we define the optimal hedging strategy of a derivative as the adjustment of the investor's optimal portfolio strategy introduced by the derivative. This leads to the following hedging results.

Proposition 4.5.1. *The hedging strategy for the portfolio $\mathcal{C}_n$ is given by*

$$\Delta_s := \frac{\Pi_s^* - \Pi_s^{0,*}}{S_s} = -h_S(s, S_s, Y_s) - \rho \frac{a(s, S_s)}{\sigma(s, S_s) S_s} h_y(s, S_s, Y_s), \quad (4.39)$$

which represents the optimal number of shares the investor should put into the traded asset due to the presence of the portfolio.

Following Musiela and Zaripholoulou (2004a), we introduce the indifference price process

$$H_s = h(s, S_s, Y_s), \quad t \le s \le T. \quad (4.40)$$

Then, the following results follows directly from the equation (4.28) and Itô calculus.

Proposition 4.5.2. *The indifference price process $H_s, s \in [t,T]$ satisfies*

$$
\begin{aligned}
H_s = & \int_t^s \left(\frac{1}{2}\gamma(1-\rho^2)a^2 h_y^2 + \rho\lambda a h_y + \mu S_u h_S \right) du + \int_t^s \sigma S_u h_S dW_u^1 \\
& + \int_t^s a h_y dW_u + h(t,S,y) - \sum_{i=1}^n c_i(S_{T_i}, Y_{T_i}) \mathbf{1}_{\{t<T_i\le s\}}.
\end{aligned} \tag{4.41}
$$

We next define the residual optimal wealth process and the residual risk process.

Definition 4.5.3. *Let X_s^*, $X_s^{0,*}$ and H_s be given, respectively, by (4.37), (4.38) and (4.41). Then we define the residual optimal wealth process by $L_s \triangleq X_s^* - X_s^{0,*}$, $s \in [t,T]$, with initial data $L_t = -h(t,S,y)$, and the residual risk process by $R_s \triangleq L_s - H_s$, $s \in [t,T]$, with initial data $R_t = 0$.*

Clearly, it follows from (4.37) and (4.38) that the process

$$
\begin{aligned}
L_s = & - h(t,S,y) + \sum_{i=1}^n c_i(S_{T_i}, Y_{T_i}) \mathbf{1}_{\{t<T_i\le s\}} \\
& + \int_t^s \mu(\Pi_u^* - \Pi_u^{0,*}) du + \int_t^s \sigma(\Pi_u^* - \Pi_u^{0,*}) dW_u^1 \\
= & - h(t,S,y) + \sum_{i=1}^n c_i(S_{T_i}, Y_{T_i}) \mathbf{1}_{\{t<T_i\le s\}} \\
& - \int_t^s \left(S_u h_S + \rho \frac{a}{\sigma} h_y \right) \left(\mu du + \sigma dW_u^1 \right),
\end{aligned} \tag{4.42}
$$

where μ, σ depend on (u, S_u), a on (u, Y_u) and h on (u, S_u, Y_u). Further, combining (4.41) and (4.42) yields the dynamics of R_s:

$$
\begin{aligned}
R_s \;:=\; & L_s + H_s \\
= \; & \int_t^s \frac{1}{2}\gamma(1-\rho^2)a^2 h_y^2 du + \int_t^s a h_y d(W_u - \rho W_u^1) \\
= \; & \int_t^s \frac{1}{2}\gamma(1-\rho^2)a^2 h_y^2 du + \sqrt{1-\rho^2}\, a h_y dW_u^{\perp},
\end{aligned} \tag{4.43}
$$

where $W_s^{\perp} = (\rho W_s^1 - W_s)/\sqrt{1-\rho^2}$ is a Brownian motion orthogonal to W^1. To this end, we have the following decomposition result.

Theorem 4.5.4. *The payoffs of the portfolio $\mathcal{C}_n(S,Y)$ specified in (4.3) admit the* *following decomposition*

$$\begin{aligned}\sum_{i=1}^{n} c_i(S_{T_i}, Y_{T_i})\mathbf{1}_{\{t<T_i\}} =& h(t, S_t, Y_t) + \int_t^T h_S dS_s + \int_t^T \rho\frac{a}{\sigma}h_y\frac{dS_s}{S_s} \\ &+ \frac{1}{2}\gamma(1-\rho^2)\int_t^T a^2h_y^2 ds + \sqrt{1-\rho^2}\int_t^T ah_y dW_s^{\perp},\end{aligned} \tag{4.44}$$

where $h(t, S, y)$ is the pricing function given by (4.28), $\sigma(s, S_s)$ and $a(s, Y_s)$ are the volatilities of S_s and Y_s, respectively.

Proof. It follows from (4.42) that the residual wealth at time T^+ is

$$L_{T^+} = -H_t - \int_t^T \left(S_s h_S(s, S_s, Y_s) + \rho\frac{a}{\sigma}h_y(s, S_s, Y_s)\right)\frac{dS_s}{S_s} + \sum_{i=1}^{n} c_i(S_{T_i}, Y_{T_i})\mathbf{1}_{\{t<T_i\}}.$$

By the residual risk process (4.43), we have

$$L_{T^+} + H_{T^+} = \frac{1}{2}\gamma(1-\rho^2)\int_t^T a^2h_y^2(s, S_s, Y_s)ds + \sqrt{1-\rho^2}\int_t^T ah_y(s, S_s, Y_s)dW_s^{\perp}.$$

Thus, the decomposition (4.44) follows from the last two equations since $H_{T^+} = 0$. □

Intuitively, the derivative payoff is decomposed into three components: the indifference price, the hedgeable risk, and the residual risk. The hedgeable risk, which is characterized by the second and the third terms in (4.44), is captured by the proceeds from the trading in S. Indeed, the integrand of the second term reflects the hedge of the risk associated with the tradable asset S, which corresponds to the delta hedge of the Black-Scholes model. The integrand of the third term represents the amount of staking in the tradable asset S for hedging the risk ingredient generated by the nontraded asset Y. With no surprise, we observe a distortion factor because of the non-prefect correlation between S and Y. When $\rho = 0$, namely the nontraded asset Y is uncorrelated to the tradable one S, the dynamics of S contains no information about the fluctuation of Y. Thus trading in S does not provide any benefit for hedging the risk associated with Y. When $\rho \to 1$ and $\frac{b}{a} = \frac{\mu}{\sigma}$, the integrand of the third term becomes the delta hedge of the Black-Scholes model. Finally, the last two terms in (4.44) represent the residual risk that is accumulated over time.

4.6 Relative Indifference Price

In this section, we consider the relative indifference price for the incomplete market setting specified in the previous section. Generally speaking, the relative price is defined as the amount of money that compensates exactly a utility maximizer with ex-ante risk exposure to take the risk of holding an additional derivative. This pricing concept was first introduced by Musiela and Zariphopoulou (2001b) and was further developed in a stochastic volatility model by Stoikov (2005).

Definition 4.6.1. *Let $V^{\mathcal{C}_k}(t,S,x,y)$ denote the buyer's value function for holding the portfolio $\mathcal{C}_k$. Then the relative indifference price for buying an additional claim with the payoff $c_{k+1}(S_{T_{k+1}}, Y_{T_{k+1}})$ and maturity $T_{k+1} \leq T$, given holding ex ante the portfolio $\mathcal{C}_k$, is defined by the mount of money $\varepsilon(c_{k+1}|\mathcal{C}_k;\gamma)$ such that*

$$V^{\mathcal{C}_k}(t,S,x,y) = V^{\mathcal{C}_{k+1}}(t,S,x-\varepsilon(c_{k+1}|\mathcal{C}_k;\gamma),y), \tag{4.45}$$

for $(t,S,x,y) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$, where the value function $V^{\mathcal{C}_n}$ is defined in (4.7).

Conceptually, this relative price generalizes the classical indifference price by embedding the preexistent risk exposure into the derivative price. Without the presence of any ex-ante risk, the relative price reduces to the classical sense. It thus follows immediately from the above definition that $\nu(c_{k+1};\gamma) = \varepsilon(c_{k+1}|\mathcal{C}_0;\gamma)$. The following results characterize the additivity of relative indifference price.

Proposition 4.6.2. *The relative indifference price defined by (4.45) is given by*

$$\varepsilon(c_{k+1}|\mathcal{C}_k;\gamma) = \nu(\mathcal{C}_{k+1};\gamma) - \nu(\mathcal{C}_k;\gamma), \tag{4.46}$$

where $\nu(\mathcal{C}_k;\gamma)$ is the classical indifference price of the portfolio $\mathcal{C}_k$.

Moreover, the classical indifference price $\nu(\mathcal{C}_n;\gamma)$ of the portfolio $\mathcal{C}_n$ admits the following decomposition

$$\nu(\mathcal{C}_n;\gamma) = \sum_{k=1}^{n} \varepsilon(c_k|\mathcal{C}_{k-1};\gamma), \tag{4.47}$$

where $\varepsilon(c_1|\mathcal{C}_0;\gamma) = \nu(c_1;\gamma)$ *is the classical indifference price of claim* c_1*.*

Proof. The identity (4.46) is a direct consequence of the definitions of indifference price (4.27) and relative indifference price (4.45). The decomposition (4.47) follows by summing up (4.46) over k. □

We next use this property to derive the quasilinear PDE for the relative indifference price.

Theorem 4.6.3. *The relative indifference price* $\varepsilon(c_\tau|\mathcal{C}_n)$ *for contingent claim* $c_\tau(S_\tau, Y_\tau)$*,* $\tau \in [0,T]$*, given holding ex ante the portfolio* $\mathcal{C}_n$*, is given by*

$$\varepsilon(c_\tau|\mathcal{C}_n) = g(t,S,y),$$

where the pricing function $g : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *solves quasilinear PDE*

$$g_t + \tilde{\mathcal{L}}^{S,y} g - \gamma(1-\rho^2)a^2 h_y g_y - \frac{1}{2}\gamma(1-\rho^2)a^2 g_y^2 = -c_\tau(S,y)\delta(t-\tau), \qquad (4.48)$$

with terminal condition $g(T^+,S,y) = 0$*, where* $h(t,S,y)$ *is the indifference pricing function solving (4.28).*

The Risk Monitoring Strategy. To construct the optimal hedging strategies, let's recall from (4.36) that the optimal control process $\Pi_s^{n,*}$, $s \in [t,T]$, for portfolio $\mathcal{C}_n$ is given by

$$\Pi_s^{n,*} = \frac{1}{\gamma}\frac{\lambda(s,S_s)}{\sigma(s,S_s)} - S_s m_S(s,S_s) - S_s h_S^n(s,S_s,Y_s) - \rho\frac{a(s,S_s)}{\sigma(s,S_s)} h_y^n(s,S_s,Y_s), \quad (4.49)$$

where the superscript n denotes the number of claims involved. To this end, we have the following risk monitoring result.

Proposition 4.6.4. *The hedging strategy for the claim* C_τ *is given by*

$$\Delta_s := \frac{\Pi_s^{n+1,*} - \Pi_s^{n,*}}{S_s} = -g_S(s,S_s,Y_s) - \rho\frac{a(s,S_s)}{\sigma(s,S_s)S_s} g_y(s,S_s,Y_s). \qquad (4.50)$$

The above hedging strategy Δ_s represents the optimal number of shares the investor should adjust her holdings in the traded asset due to the presence of the additional claim C_τ.

We conclude this section by introducing the relative indifference price process

$$G_s = g(s, S_s, Y_s), \quad t \le s \le T. \tag{4.51}$$

The following results follows directly from the equation (4.48) and Itô calculus.

Proposition 4.6.5. *The relative indifference price process $G_s, s \in [t, T]$, is given by*

$$\begin{aligned} G_s = & \int_t^s \left(\frac{1}{2}\gamma(1-\rho^2)a^2 g_y^2 + \gamma(1-\rho^2)a^2 h_y g_y + \rho\lambda a g_y + \mu S_u g_S \right) du \\ & + \int_t^s \sigma S_u g_S dW_u^1 + \int_t^s a g_y dW_u + g(t, S, y) - c_\tau(S_\tau, Y_\tau)\mathbf{1}_{\{t<\tau\le s\}}, \end{aligned} \tag{4.52}$$

where $h(t, S, y)$ solves (4.28).

4.7 Foresighted Indifference Valuation

In this section, we define and construct the foresighted indifference price. We first specify the structure of investment opportunities. We consider a venture capitalist with exponential utility $u(x) = -e^{-\gamma x}, \gamma > 0$, who aims to maximize the expected utility of terminal wealth at time T. Suppose that before time $t < T$ the venture capitalist has been involved ex ante in a venture project $\mathcal{Y}$, whose value is modelled via a Itô diffusion

$$dY_s = b(s, Y_s)ds + a(s, Y_s)dW_s, \quad t \le s, \tag{4.53}$$

with $Y_t = y \in \mathbb{R}$, where coefficients b and a guarantee a unique strong solution.

As specified by the VC contract, the venture capitalist will receive the contingent payoff $C_{T_1} = c_1(Y_{T_1})$ at a given time T_1 provided the staged financing of amount K_1 at a prepecified time t_1 in the future, where $t < t_1 < T_1 \le T$. The venture capitalist can abandon the project by simply defaulting the financing at t_1, in which case the project ends and no payoff on either party. This special feature of financing arising from venture capital is called "contingent pre-contracting" (or "ex-ante staging" in Kaplan and Stömberg, 2002) between the venture capitalist and the entrepreneur in the literature contract theory.

Now, suppose that at time t the venture capitalist is offered another investment opportunity with contingent payoff $C_{T_2} = c_2(Y_{T_2})$ realized at the future time T_2 with $t_1 < T_2 \leq T$. Note that the timing relation $t < t_1 < T_1, T_2 \leq T$. We impose $t_1 < T_2$ to exclude the trivial case. No restriction is imposed on the order between T_1 and T_2.

Although no trading in $\mathcal{Y}$ occurs throughout the entire time horizon $[t, T]$, the venture capitalist can trade dynamically in the financial market with a riskless $\mathcal{B}$ bond and a risky stock $\mathcal{S}$. For simplicity, we assume constant interest $r = 0$, and the stock dynamics is modelled via a lognormal diffusion

$$dS_s = \mu S_s ds + \sigma S_s dW_s^1, \quad t \leq s, \tag{4.54}$$

with $S_t > 0$. It is also assumed that no intermediate consumption is allowed, nor is exogenous funds available for the venture capitalist. The question that we are interested in is *how to value the second claim C_{T_2}?*

Before we construct the concrete valuation model, let us emphasize the key features of this problem: (i) Investing in the second claim $c_2(Y_{T_2})$ will adjust the venture capitalist's preference on additional risk and thus may affect the her decision on whether financing the venture project C_{T_1} at time t_1. So it is obviously not optimal to value $c_2(Y_{T_2})$ in isolation. (ii) The venture capitalist actually can take advantage of the feature of "ex-ante staging" offered by the VC contract. She should embed such information about the financing of future stages into her valuation process. (iii) The standard relative indifference pricing approach introduce previously fails in this situation in that more information and optionality are available here and should leads to more value. We should not price the second claim basing only on the payoff.

Stemming from utility-indifference, the foresighted valuation method is characterized by two optimal investment problems. The first problem corresponds to disregarding the second claim $c_2(Y_{T_2})$, where the venture capitalist is faced with hedging an standard installment option. That is, the venture capitalist solves utility maximization problem

$$V^{\mathcal{I}_1}(t, x, y) := \max_{\tau \in \mathcal{T}} \sup_{\pi[t,T] \in \mathcal{A}} E\left[u\left(X_T + C_{T_1} \mathbf{1}_{\{\tau = T_1\}} \right) \big| X_t = x, Y_t = y \right], \tag{4.55}$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, where the wealth process (X_s) is given by

$$X_s = x - \sum_{i=1}^{n} K_i \mathbf{1}(t_i < \tau)\mathbf{1}(t_i < s) + \int_t^s \mu \pi_u du + \int_t^s \sigma \pi_u dW_u^1, \tag{4.56}$$

for $s \in [t,T]$, where $\tau \in \mathcal{T} := \{t_1, \cdots, t_n, T_1\}$ stands for the contract terminating time.

In the second problem, the venture capitalist take the opportunity to invest in the second claim $c_2(Y_{T_2})$. In this case, the venture capitalist's portfolio holdings include a standard installment option $\mathcal{I}_1$ plus a European option with the payoff C_{T_2}, which is can be viewed as a generalized installment option with recover payoff C_{T_2}. We denote it by $\mathcal{I}_2$. The corresponding the value function is then defined by

$$V^{\mathcal{I}_2}(t,x,y) := \max_{\tau \in \mathcal{T}} \sup_{\pi[t,T] \in \mathcal{A}} E\left[u\left(X_T + C_{T_1}\mathbf{1}_{\{\tau=T_1\}} + C_{T_2}\right)\big| X_t = x, Y_t = y\right], \tag{4.57}$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, where the wealth process (X_s) is given by (4.56).

We next introduce the following definition of foresighted indifference price.

Definition 4.7.1. *The foresighted indifference price of claim C_{T_2} is defined as the amount of money ε that compensates exactly the utility maximizer to take the risk of holding an additional derivative C_{T_2} conditional on holding ex ante the installment option $\mathcal{I}_1$ with the payoff C_{T_1}. That is, $\varepsilon(C_{T_2}|\mathcal{I}_1)$ satisfies the indifference condition*

$$V^{\mathcal{I}_2}(t,x,y) = V^{\mathcal{I}_2}(t, x - \varepsilon(C_{T_2}|\mathcal{I}_1), y) \tag{4.58}$$

for $(t,x,y) \in [t,T] \times \mathbb{R} \times \mathbb{R}$, where $V^{\mathcal{I}_2}$ and $V^{\mathcal{I}_2}$ are defined by (4.57) and (4.57), respectively.

The following property follows immediately from the above definition.

Proposition 4.7.2. *The foresighted indifference price defined by (4.58) is given by*

$$\varepsilon(C_{T_2}|\mathcal{I}_1) = \phi(\mathcal{I}_2) - \phi(\mathcal{I}_1), \tag{4.59}$$

where $\phi(\mathcal{I}_i)$ are the indifference price for installment options $\mathcal{I}_i$, $i = 1, 2$.

I what follows, we derive a formula for the foresighted indifference price. We start by constructing the associated value functions. To remain consistent with the discussion in the previous chapters, we adopt the notation an accumulated portfolio

$$G_k \triangleq \{c_1(Y_{T_1}), \cdots, c_k(Y_{T_k})\},$$

for $k = 0, 1, 2$ in this simple setting, where $G_0 \equiv 0$ denotes no portfolio holdings.

Then following the analysis in the last chapter, we derive the value function $V^{\mathcal{I}_2}$.

Proposition 4.7.3. *The value function (4.57) for buying the installment option $\mathcal{I}_2$ with recover payoff is given by*

$$V^{\mathcal{I}_2}(t,x,y) = -\exp\left\{-\gamma\left(x + \frac{1}{2\gamma}\frac{\mu^2}{\sigma^2}(T-t) + \phi(\mathcal{I}_2)\right)\right\}, \tag{4.60}$$

for $(t,x,y) \in [t_1,t_2) \times \mathbb{R} \times \mathbb{R}$, where $\phi(\mathcal{I}_2)$ is the indifference price of installment option $\mathcal{I}_2$ defined by the function

$$\phi(t,y) = \frac{1}{\gamma\bar{\rho}^2}\ln\left(E_{\mathbb{Q}}^{t,y}\left[\exp\left\{\gamma\bar{\rho}^2\left([\nu_{t_1}(C_{T_1}|C_{T_2}) - K_1]^+ + \nu_{t_1}(C_{T_2})\right)\right\}\right]\right), \tag{4.61}$$

for $(t,y) \in [t_1,t_2] \times \mathbb{R}$, where $\nu(\cdot|\cdot)$ is relative indifference pricing functional and $\bar{\rho} = \sqrt{1-\rho^2}$.

Obviously, by letting $C_{T_2} = 0$ in we recover the value function $V^{\mathcal{I}_1}$. Therefore, the following pricing formula follows from the definition of foresighted indifference price.

Theorem 4.7.4. *The foresighted indifference price defined by (4.58) is given by*

$$\varepsilon(t,y) = \frac{1}{\gamma\bar{\rho}^2}\ln\frac{E_{\mathbb{Q}}^{t,y}\left[\exp\left\{\gamma\bar{\rho}^2\left([\nu_{t_1}(C_{T_1}|C_{T_2}) - K_1]^+ + \nu_{t_1}(C_{T_2})\right)\right\}\right]}{E_{\mathbb{Q}}^{t,y}\left[\exp\left\{\gamma\bar{\rho}^2\left([\nu_{t_1}(C_{T_1}) - K_1]^+\right)\right\}\right]}, \tag{4.62}$$

for $(t,y) \in [t,t_1] \times \mathbb{R}$, where $\bar{\rho} = \sqrt{1-\rho^2}$.

We conclude the discussion of foresighted valuation by investigating some special cases of the pricing function (4.62).

- If both $[\nu_{t_1}(C_{T_1}|C_{T_2}) - K_1]^+$ and $[\nu_{t_1}(C_{T_1}|G_0) - K_1]^+$ are equal to 0, then $\varepsilon(C_{T_2}) = \nu_{t_1}(C_{T_2}|G_0)$ is the relative indifference price of C_{T_2} conditional on holding $G_0 = 0$. This is the case when the venture capitalist will not financing the VC project wether or not buying the second claim C_{T_2} (or, the investor has no future investment opportunity).

- If both $[\nu_{t_1}(C_{T_1}|C_{T_2})-K_1]^+$ and $[\nu_{t_1}(C_{T_1}|G_0)-K_1]^+$ are strictly positive, then $\varepsilon(C_{T_2}) = \nu_{t_1}(C_{T_2}|G_1)$ is the relative indifference price of C_{T_2} conditional on holding G_0 plus C_{T_1}. This is the case when the investor will definitely finance the VC contract at time t_1 whether or not buy C_{T_2}.

- If $[\nu_{t_1}(C_{T_1}) - K_1]^+ = 0$ and $\nu_{t_1}(C_{T_2}) = 0$, then

$$\varepsilon(t, y) = \frac{1}{\gamma(1-\rho^2)} \ln E_{\mathbb{Q}} \left[\exp\left\{ \gamma(1-\rho^2)[\nu_{t_1}(C_{T_1}|G_0) - K_1]^+ \right\} \big| Y_t = y \right],$$

corresponding to the utility indifference price of an installment option with installment K_2 and payoff g_2.

Chapter 5

Indifference Valuation and Risk Management of Installment Options

5.1 Introduction

In this paper we present a utility-based approach to price and hedge installment options in an incomplete market environment. The installment option is a financial innovation that allows investors to gain direct exposure to the stock market by paying a fraction of the share price upfront and delaying the rest of optional payments until later dates. This high gearing instrument has two characteristics that differentiate it from a conventional option: (i) the option premium is paid by installments – scheduled periodically over the life of the option – instead of a lump sum upfront; and (ii) the holder has the right to terminate the contract by defaulting any installment payment, which leaves no further liability to either party.

One example of such products is the installment warrants listed on the Australian Stock Exchange (ASX). Since they were first launched on the market in 1997, installment warrants have achieved significant growth (as illustrated in Figure 5.1).[1] Another successful product is the so called FX installment options, which are written on foreign exchange rates and traded actively on European over-the-counter

[1] For the detailed historical record of ASX installments, we refer the reader to http://www.asx.com.au.

(OTC) markets.[2] For a brief introduction and examples of FX installments, we refer to [135].

The popularity of installment options can be attributed to its various attractive features that ease the investor's portfolio management. In particular, it reduces significantly the initial premium, though it entitles the holders to all the benefits of the underlying security such as capital growth and dividends. This low cost of investment facilitates the investor's budget schedule whilst increases the leverage level. Moreover, the optionality of termination without penalty prevents potential losses and reduces the liquidity risk, typically associated with other OTC derivatives. This also enables investors to acquire more information for decision making so as to improve their overall investment optimality. Finally, it is helpful in many situations where necessity of hedging is uncertain [see 135].

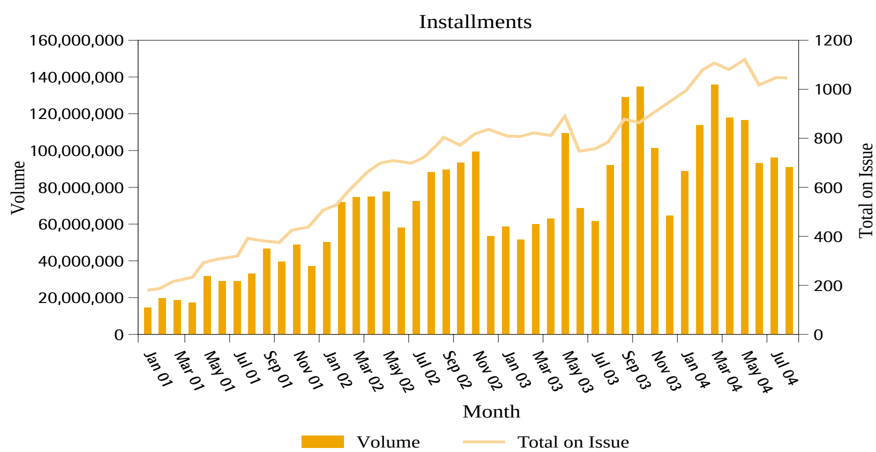


Figure 5.1: The market volume of installment warrants in ASX

The valuation of installment options has become an active research area since [36]. They derived bounds for the value of an installment option using no-arbitrage principle and investigated the static versus dynamic hedging strategies in a stochastic volatility model. In a similar setting, [37] studied further the static hedging strategy. Recently, more research on installment options has been done in the Black-Scholes setting. [11] developed a dynamic programming procedure to price installment call and put options with application to ASX installments.

[2]See, for example, the X-markets of Deutsche Bank for a detail description of such products.

[135] compared numerical techniques for pricing FX installment options written on exchange rates. Finally, [24] derived an inhomogeneous Black-Scholes PDE for pricing American continuous-installment calls and puts, and the associated pricing formulae using the integral representation method. To the best of our knowledge, the literature is currently restricted in the arbitrage-free framework. In the present paper we examine the valuation and hedging of installment options in incomplete market environments.

We apply the so called utility-based approach, according to which the derivative price is defined as the amount that compensates an optimally behaving investor for taking the risk of holding the derivative. In other words, the indifference price would make the investor indifferent between buying the option or not. This pricing concept was used originally by [72] in analyzing the impact of transaction costs on the price. It has been extensively studied in the literature. We refer the reader to, among others, [118, 40, 111], and [83].

Herein, we study new applications of the utility-based valuation in the pricing and hedging of installment options. Their payoffs are contingent on tradable securities and non-traded risk factors, modelled as correlated diffusion processes. By using dynamic programming techniques, we solve the underlying stochastic optimization problems and characterize the indifference price using the associated Hamilton-Jacobi-Bellman (HJB) equations. We derive a second order quasilinear PDE that the price satisfies. We further investigate the hedging strategy, payoff decomposition, and probabilistic representation of the indifference price. It turns out that, under certain modeling assumptions, the value of installment options can be represented by a recursive formula involving a nonlinear expectation. This model restriction provides a more tractable framework that is convenient in analyzing venture capital and sequential investments, as discussed in Section 5.6. In addition, we develop a Monte-Carlo procedure to analyze numerical examples arising from ASX installment warrants and venture capital financing, where we adopt the least-squares method (LSM), originally proposed by [92].

Besides the contribution to installment options, our work also provides new insights and methodological advances to other applied problems, such as *compound options*, *sequential investments*, and *venture capital*. Indeed, the installment option is closely related to a compound option which is an option on an option. For example, the case with two installments reduces to a compound call; see discussion in

Section 5.5. The research of compound options can be dated back at least to [14], who suggested to view the common stock as an option on the assets of the firm when valuing corporate liabilities. From this perspective, a call option on equity is a compound call on the value of the firm. Borrowing this idea, [61] valued coupon bonds using a compound option model, and later [see 62], he derived a formula for pricing a compound call on a European call. The pricing formulae for the other popular combinations were provided by Rubinstein [119]. [127] further investigated the valuation techniques arising from the pricing formula of compound options. Although the compound option model extends the Black-Scholes theory, it conceptually introduces the market incompleteness which arises from the fact that the value of the firm is a non-traded asset. The above valuation models, however, did not address the incompleteness. We tackle this issue by introducing explicitly an unhedgeable risk factor into our model.

[14] also pointed out that, when the firm has coupon bonds rather than pure discount bonds outstanding, the common equity can be viewed as a (multi-) compound option, namely, "an option on an option on $\cdots$ an option on the firm". In the present paper we attempt to implement this idea using the notion of installment option. Meanwhile, we argue that in the incomplete market framework it is more meaningful to consider an installment option rather than a compound option. The reason lies in the fact that the payoff of a generic "compound option" as a function of its underlying option is not well defined *a priori*; remember that the theoretical price of the underlying option is not unique in incomplete markets. Our model differentiates itself from the literature of installment and compound options in the modeling of market incompleteness.

As mentioned previously, the notion of installment option can be applied to value sequential investments. It is well known, in real options context, that any investment opportunity with payoffs depending on underlying assets can be viewed as an option. A standard reference on this field is [43]. We observe that many of these investments in reality have a sequential nature, where staged financing is allowed with intermediate abandoning option. In such investments, the future ventures are available only if the earlier opportunities are undertaken. It is natural to value such investments using an installment option model that incorporates unhedgeable risk.

In particular, the staged financing is a widely used technique in venture capital investments such as high-tech startups. Because of the high failure risk of a venture project, venture capitalists typically retain the option to abandon the venture whenever the project becomes nonprofitable. Such financing tools were found to be the most potent control mechanism that the venture capitalist can employ [see 120]. Financing rounds are usually set to the significant stages of the project developing process. If at any of these stages the project fails to meet the prespecified target (milestone), the venture capitalist has the right to liquidate the project. Such covenant is called "contingent pre-contracting" (or "ex-ante staging"). We refer to [81] for some empirical evidence of the covenants in the real-world venture contracts. In the contract theory framework, [27] showed that the "contingent pre-contracting" for follow-up rounds is theoretically better than the simple "right of refusal", which is an informal commitment of additional funds as needed.

Given this staging feature, the model of installment options can serve as an important building block in the valuation of venture capital investments. This modeling approach was previously considered in [38]. They derived the no-arbitrage bounds on the amount of initial funding for an imitated venture project (*clone*) under the assumption that the European options are traded for an existing firm (*innovator*) in the same industry. However, we argue that it is more straightforward to model venture projects as non-traded assets and tackle the valuation problem in an incomplete market setting. The model presented in this paper can be applied directly to analyze the staged financing of venture capital.

The paper is organized as follows. In Section 5.2, we introduce the installment option and the underlying optimal investment problems. In Section 5.3, we construct the indifference price equations when installments incorporate both tradable securities and non-traded risk factors, and discuss the properties of the indifference price. Section 5.4 provides the hedging and risk monitoring strategies. We then investigate the special case of installments contingent on a non-traded risk factor in Section 5.5. In Section 5.6, we develop a Monte Carlo procedure to compute the prices, and illustrate the application of the installment model to analyze the ASX installment warrants and the staged financing of venture capital. In Section 5.7, we provide conclusions and direction for future research.

5.2 The Market and Investment Models

We consider a continuous-time financial market with a riskless bond B and two risky assets, a tradable stock S and a non-traded asset (risk factor) Y. The stock price S is modelled as a diffusion process satisfying

$$dS_s = \mu(s, S_s)S_s ds + \sigma(s, S_s)S_s dW_s^1, \quad t \leq s, \tag{5.1}$$

with $S_t = S > 0$. The level of the non-traded asset Y is given by a correlated diffusion

$$dY_s = b(s, Y_s)ds + a(s, Y_s)(\rho dW_s^1 + \bar{\rho} dW_s^{\perp}), \quad t \leq s, \tag{5.2}$$

with $Y_t = y \in \mathbb{R}$, where $\rho \in (-1, 1)$ is the correlation coefficient and $\bar{\rho} = \sqrt{1-\rho^2}$.

The processes $(W_t^1, W_t^{\perp}; t \geq 0)$ are independent standard Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by $(W^1, W^{\perp})$ and satisfies the usual conditions. The coefficients $\mu(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $a(\cdot,\cdot)$ are assumed to satisfy all the regularity conditions such that equations (5.1) and (5.2) have a unique strong solution satisfying $S_s > 0$ $\mathbb{P}$-a.s. for a.e. $s \in [t, T]$. We assume that the riskless bond with matures at T and is available for trading over the time horizon $[t, T]$, yielding constant interest rate $r = 0$. The case $r > 0$ can be treated using standard rescaling arguments.

Now, suppose that an installment option, denoted by $\mathcal{I}_n(S, Y)$, is introduced at time t. Its payoff is given by $G_T = g(S_T, Y_T)$ with the payoff function $g(S, y)$ assumed to be bounded. To receive this payoff, the buyer has to pay an upfront premium ν at time t, and n optional installments $K_n, \cdots, K_1$ at the installment dates $t_n, \cdots, t_1 \in (t, T)$, respectively. The holder, however, has the right to stop making the installment payment at any installment date t_i, thereby terminating the contract on the due date of the first defaulted payment. For convenience, we label the installment time in such a way that $0 = t_{n+1} < t_n < \cdots < t_1 < t_0 = T$; see Figure 5.2 for an illustration of such installment structure.

The valuation problem is to determine the holding value ν (upfront value) of such a contract (at time t) as well as the associated hedging strategy given the installment structure. Note that, in our model, the installments $K_i = K_i(S_{t_i}, Y_{t_i})$

can be $\mathcal{F}_{t_i}$-measurable random variables rather than constants. This flexibility offers an important advantage when analyzing venture capital and other sequential investments.

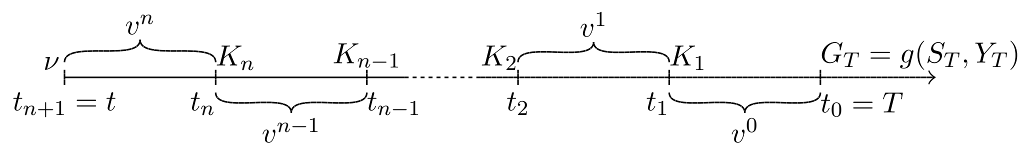


Figure 5.2: An installment option $\mathcal{I}_n$ with n optional installments $K_n, \cdots, K_1$.

The indifference valuation methodology is based on the comparison of the optimal investment opportunities with and without the installment option. In both situations, dynamic trading between the bond and the stock occurs over the time horizon $[t, T]$. Throughout, it is assumed that no exogenous funds are available, nor is intermediate consumption allowed. The investor uses a self-financing strategy to rebalance his/her portfolio allocations. Starting at time t with initial wealth $x \in \mathbb{R}$, the investor holds Δ_s shares of stock for $s \in [t, T]$ with the (spot) value of $\pi_s = \Delta_s S_s$, and deposits the remainder of his/her wealth into the bond account. It follows that the total current wealth X_s^0, in the absence of installments, solves

$$dX_s^0 = \mu(s, S_s)\pi_s ds + \sigma(s, S_s)\pi_s dW_s^1, \tag{5.3}$$

for $s \in [t, T]$, with initial condition $X_t^0 = x \in \mathbb{R}$ [see 96]. The set of admissible policies is defined by $\mathcal{A}_{[t,T]} := \big\{\pi : \pi_s \text{ is } \mathcal{F}_s \text{ predictable and } E\int_t^T \sigma^2(s, S_s)\pi_s^2 ds < \infty\big\}$.

The next step is to introduce stochastic optimization problems for analyzing the investor's optimal behavior, through which the indifference price will be constructed. Throughout the analysis, we concentrate on a *constant absolute risk aversion* (CARA) investor with utility function given by

$$u(x) = -e^{-\gamma x}, \quad x \in \mathbb{R}, \tag{5.4}$$

where $\gamma > 0$ is the risk aversion parameter. The investor's objective is to maximize the expected utility of terminal wealth at T.

The first optimization problem is the classical Merton model of optimal investment, where the investor seeks to maximize the terminal expected utility without holding the installment option. The value function is defined as

$$M(t,S,x) := \sup_{\pi\in\mathcal{A}_{[t,T]}} E\left[u(X_T^0)\middle|S_t=S, X_t^0=x\right], \tag{5.5}$$

for $(t,S,x)\in[0,T]\times\mathbb{R}^+\times\mathbb{R}$, with terminal condition $M(T,S,x)=u(x)$, where the wealth process (X_s^0) satisfies (5.3) and $\mathcal{A}_{[t,T]}$ is the set of admissible policies.

In the second problem, the installment option $\mathcal{I}_n(S,Y)$ is incorporated. We assume that it is bought at time $t<\min_{1\le i\le n}T_i$. Recall that in $[t,T]$ the factor Y is not traded. The option holder has to pay the installment K_i at each installment date t_i until the discretionary time τ at which s/he decides to terminate the contract. Hence, the wealth dynamics suffer a downside jump of size K_i at each $t_i<\tau$, though they are still governed by (5.3) between two successive installment dates. The wealth process X_s for holding the installment option is given by

$$X_s = x-\sum_{i=1}^{n}K_i\mathbf{1}_{\{t<t_i<s\wedge\tau\}}+\int_t^s\mu\pi_u du+\int_t^s\sigma\pi_u dW_u^1, \tag{5.6}$$

for $s\in[t,T]$, where $\tau\in\mathcal{T}:=\{t_n,\cdots,t_1,t_0=T\}$ denotes the contract terminating time. The corresponding value function is then defined by

$$V^n(t,S,x,y) := \sup_{\tau\in\mathcal{T},\pi\in\mathcal{A}_{[t,T]}} E\left[u\left(X_T+G_T\mathbf{1}_{\{\tau=T\}}\right)\middle|S_t=S,X_t=x,Y_t=y\right], \tag{5.7}$$

for $(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}$, where X_s is the wealth process given by (5.6) and $\mathcal{A}_{[t,T]}$ as defined previously.

5.3 Utility-based Valuation

In this section we construct the indifference price of the installment option $\mathcal{I}_n(S,Y)$. The definition of indifference price for claims involving early exercise was first introduced by [39] for models with transaction costs, and later by [110] for models with a non-traded asset. The following definition is a direct extension of the latter.

Definition 5.3.1. *The buyer's indifference price of the installment option $\mathcal{I}_n(S,Y)$ with terminal payoff $G_T=g(S_T,Y_T)$ and n installments – as specified in the previous* *section – is defined as the amount $\nu(\mathcal{I}_n; \gamma, t)$ such that*

$$M(t, S, x) = V^n(t, S, x - \nu(\mathcal{I}_n; \gamma, t), y), \tag{5.8}$$

for $(t, S, x, y) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, where the value functions M and V^n are defined in (5.5) and (5.7), respectively.

To determine the indifference price ν, we first need to study the value functions M and V^n. A standard approach is to study the associated Hamilton-Jacobi-Bellman (HJB) equations. For ease of presentation, we suppress the arguments of the market coefficients and introduce the differential operators

$$\mathcal{L}^{(S)} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \mu S \frac{\partial}{\partial S}, \tag{5.9}$$

$$\mathcal{L}^{(S,y)} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma a S \frac{\partial^2}{\partial S \partial y} + \frac{1}{2}a^2 \frac{\partial^2}{\partial y^2} + \mu S \frac{\partial}{\partial S} + b \frac{\partial}{\partial y}, \tag{5.10}$$

$$\tilde{\mathcal{L}}^{(S,y)} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma a S \frac{\partial^2}{\partial S \partial y} + \frac{1}{2}a^2 \frac{\partial^2}{\partial y^2} + \left(b - \rho\frac{\mu}{\sigma}a\right) \frac{\partial}{\partial y}. \tag{5.11}$$

Note that $\mathcal{L}^{(S)}$ and $\mathcal{L}^{(S,y)}$ are the infinitesimal generators of Markov process S and (S, Y), respectively. To this end, we apply dynamic programming and stochastic calculus to derive the HJB equation for the Merton problem, which is given by

$$M_t + \max_{\pi} \left(\frac{1}{2}\sigma^2 \pi^2 M_{xx} + \pi \left(\sigma^2 S M_{Sx} + \mu M_x \right) \right) + \mathcal{L}^{(S)} M = 0, \tag{5.12}$$

for $(t, S, x) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}$, with terminal condition $M(T, S, x) = -e^{-\gamma x}$. It turns out that this equation can be linearized and solved by Feynman-Kač connection. We summarize the result as follows (the proof can be found in Appendix).

Proposition 5.3.2. *The value function M defined in (5.5) is given by*

$$M(t, S, x) = -\exp\{-\gamma(x + m(t, S))\}, \tag{5.13}$$

with

$$m(t, S) = \frac{1}{\gamma} E_{\mathbb{Q}} \left[\frac{1}{2} \int_t^T \frac{\mu^2(s, S_s)}{\sigma^2(s, S_s)} ds \Big| S_t = S \right], \tag{5.14}$$

where $\mathbb{Q}$ is the martingale measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\int_0^T \frac{\mu(s,S_s)}{\sigma(s,S_s)}dW_s^1 - \frac{1}{2}\int_0^T \frac{\mu^2(s,S_s)}{\sigma^2(s,S_s)}ds\right\}. \tag{5.15}$$

The above pricing measure $\mathbb{Q}$ is known as the *minimal martingale measure*, under which the discounted traded asset is a martingale while the law of the orthogonal martingale measure remains unchanged. This measure was originally introduced by [52]. For models with continuous price process, it is the martingale measure minimizing the reverse entropy $H(\mathbb{P}|Q)$ over all local martingale measures $Q \in \mathcal{M}$; see [125, 126]. We recall that the *relative entropy* $H(Q|P)$ of any probability measure Q with respect to P is defined as

$$H(Q|P) := \begin{cases} E\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P. In the present model, where the market coefficients μ and σ depend only on (t, S_t), the minimal martingale measure $\mathbb{Q}$ coincides with the *minimal entropy martingale measure*. The latter is the martingale measure minimizing the relative entropy $H(Q|\mathbb{P})$ over $Q \in \mathcal{M}$. For further discussions, we refer to [54, 118], and [40]. In addition, the dynamics of (S, Y) under $\mathbb{Q}$ become

$$dS_s = \sigma(s,S_s)S_s d\tilde{W}_s^1, \tag{5.16}$$

$$dY_s = \left(b(s,Y_s) - \rho\frac{\mu(s,S_s)}{\sigma(s,S_s)}a(s,Y_s)\right)ds + a(s,Y_s)\left(\rho d\tilde{W}_s^1 + \bar{\rho}dW_s^{\perp}\right). \tag{5.17}$$

To construct the value function V^n (cf. (5.7)), we start from the last period where no installment is left to pay (say, $n = 0$). In this case, the optimization problem reduces to that involving a traditional European option. [111] studied this problem assuming lognormal stock dynamics, in which the argument S vanishes away from the value function. In the present setting, the associated HJB equation turns out to be

$$V_t + \max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2V_{xx} + \pi\left(\sigma^2 SV_{Sx} + \rho\sigma aV_{xy} + \mu V_x\right)\right) + \mathcal{L}^{(S,y)}V = 0, \tag{5.18}$$

for $(t,S,x,y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, with terminal condition

$$V(T,S,x,y) = -\exp\{-\gamma(x+g(S,y))\},$$

where the operator $\mathcal{L}^{(S,y)}$ is defined in (5.10). The value function V is the unique viscosity solution of (5.18) and (5.19) in the class of functions that are concave and nondecreasing in x and uniformly bounded in y [see 110, 46]. We further have the following results, the proof of which is provided in Appendix.

Proposition 5.3.3. *The value function V^0 for the European payoff $G_T = g(S_T, Y_T)$ is given by*

$$V^0(t,S,x,y) = -\exp\left\{-\gamma(x+m(t,S)+h(t,S,y))\right\}, \tag{5.19}$$

for $(t,S,x,y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, where the functions m is given by (5.14) and h solves the quasilinear equation

$$h_t + \tilde{\mathcal{L}}^{(S,y)}h - \frac{1}{2}\gamma(1-\rho^2)a^2h_y^2 = 0, \tag{5.20}$$

for $(t,S,y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$, with terminal condition $h(T,S,y) = g(S,y)$.

We note that, although the HJB equation (5.18) can be reduced to the quasilinear equation (5.20) using a simple log transformation, the latter cannot be further linearized due to the dimensionality and the nature of the nonlinearity. It is, however, easy to show that h is the indifference pricing function of the European payoff G_T, by applying the indifference condition (5.8) to the value functions (5.13) and (5.19). In addition, we observe that the function $V^0(t,S,\cdot,y)$ is strictly increasing for any $(t,S,y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$.

We next consider an installment option $\mathcal{I}_1(S,Y)$ with a sole outstanding installment K_1, to be paid at time $t_1 \in (t,T)$. In this simple case, $n=1$ and $\tau \in \mathcal{T} = \{t_1, T\}$. At the installment date t_1, the investor having wealth X_{t_1} – determined by the control policy π up to t_1 – is faced with the choice between terminating the contract and paying the installment K_1. If s/he choose to default the payment K_1 (in which case, $\tau = t_1$), thereby terminating the contract at t_1, then, thereafter, the investor trades exclusively between the two tradable assets. In this case, no derivative payoff will be involved at the terminal time T. The investor then solves the classical Merton problem with initial wealth X_{t_1} at time t_1 and has maximal utility given by $M(t_1, S_{t_1}, X_{t_1})$.

Alternatively, the investor can choose to pay the installment K_1 for keeping the right to exercise the option at T, which corresponds to $\tau = T$. In this case, the buyer's wealth state at t_1 becomes $X_{t_1} - K_1$. The valuation problem reduces to the optimal investment problem involving a European claim with payoff G_T, namely, $\mathcal{I}_0$. Hence the continuation value is $V^0(t_1, S_{t_1}, X_{t_1} - K_1, Y_{t_1})$. The indifference pricing condition (cf. (5.8)) further leads to

$$V^0(t_1, S_{t_1}, X_{t_1} - K_1, Y_{t_1}) = M(t_1, S_{t_1}, X_{t_1} + H_{t_1} - K_1),$$

with $H_{t_1} = h(t_1, S_{t_1}, Y_{t_1})$ being the indifference price of the European payoff G_T at t_1.

Having the right to decide whether to stop or to continue, the buyer's expected utility payoff is given by[3]

$$\begin{aligned} J^\pi(t, S, x, y) &= E_t^{S,x,y}\left[\max\left\{M(t_1, S_{t_1}, X_{t_1}), M(t_1, S_{t_1}, X_{t_1} + H_{t_1} - K_1)\right\}\right] \\ &= E\left[M\left(t_1, X_{t_1} + (H_{t_1} - K_1)^+\right) \middle| S_t = S, X_t = x, Y_t = y\right], \end{aligned}$$

for $t < t_1$ and a given control policy $\pi \in \mathcal{A}_{[t,t_1]}$ up to time t_1. Note that the monotonicity of M with respect to the spatial argument x has been used for the last equality. Moreover, it is optimal to terminate the contract if

$$M(t_1, S_{t_1}, X_{t_1}) \geq M(t_1, S_{t_1}, X_{t_1} + H_{t_1} - K_1),$$

and to continue otherwise. The monotonicity of M further implies that the optimal stopping condition is equivalent to $H_{t_1} \leq K_1$.

We easily deduce that the value function of the installment option $\mathcal{I}_1$ at time $t < t_1$ is given by

$$\begin{aligned} V^1(t, S, x, y) &= \sup_{\pi \in \mathcal{A}_{[t,t_1]}} J^\pi(t, x, y) \\ &= \sup_{\pi \in \mathcal{A}_{[t,t_1]}} E_t^{S,x,y}\left[M\left(t_1, S_{t_1}, X_{t_1} + (H_{t_1} - K_1)^+\right)\right]. \end{aligned} \tag{5.21}$$

[3]The value of the portfolio at time t_1 is $J^\pi(t_1, S_{t_1}, X_{t_1}, Y_{t_1}) = M(t_1, S_{t_1}, X_{t_1} + (H_{t_1} - K_1)^+)$ for a given control policy $\pi \in \mathcal{A}_{[t,t_1]}$.

This resembles the valuation of a European claim with the payoff $(H_{t_1} - K_1)^+$ realized at time $t_1 \in (t, T)$ for an investor with utility function $u(x) = M(t_1, S, x)$. These observations lead to the following result.

Proposition 5.3.4. *The indifference price $\nu(\mathcal{I}_1; \gamma, t)$ of an installment option, with a sole outstanding installment K_1 to be paid at $t_1 \in (t, T)$, is given by*

$$\nu(\mathcal{I}_1; \gamma, t) = v(t, S, y),$$

where the function v solves the quasilinear equation

$$v_t + \tilde{\mathcal{L}}^{(S,y)} v - \frac{1}{2}\gamma(1-\rho^2)a^2 v_y^2 = 0, \tag{5.22}$$

for $(t, S, y) \in [0, t_1] \times \mathbb{R}^+ \times \mathbb{R}$, with terminal condition

$$v(t_1, S, y) = (h(t_1, S, y) - K_1(S, y))^+.$$

Proof. From (5.21) it is clear that V^1 satisfies the HJB equation (5.18) for $(t, S, x, y) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ with terminal condition $V^1(t_1, S, x, y) = M(t_1, S, x + (h(t_1, S, y) - K_1)^+)$. Following the arguments used in Proposition 5.3.3 with payoff function $g(S, y) = (h(t_1, S, y) - K_1)^+$, we find that

$$V^1(t, S, x, y) = -\exp\left\{-\gamma(x + m(t, S) + v(t, S, y))\right\},$$

where v satisfies (5.22) with $v(t_1, S, y) = (h(t_1, S, y) - K_1)^+$. Finally, combining the indifference condition (5.8), we conclude that v is the indifference price of $\mathcal{I}_1$. $\square$

We next apply the backward induction arguments to derive the pricing equation for a generic installment option $\mathcal{I}_n(S, Y)$ with n installments $K_n, \cdots, K_1$, scheduled at dates $t_n, \cdots, t_1$, respectively. We remind the reader that $0 = t_{n+1} < t_n < \cdots < t_1 < t_0 = T$ (see Figure 5.2). To this end, let $v^i(t, y)$ denote the indifference price of the installment option with i outstanding installments for $t \in [t_{i+1}, t_i]$. By Theorem 5.3.4 and backward induction, we see that the pricing function $v^n(t, S, y)$ solves the equation (5.22) on $[0, t_n] \times \mathbb{R}^+ \times \mathbb{R}$ with terminal condition $v^n(t_n, S, y) = (v^{n-1}(t_n, S, y) - K_n)^+$. The results are summarized below.

Theorem 5.3.5. *The indifference price $\nu(\mathcal{I}_n;\gamma,t)$ defined in (5.8) is given by*

$$\nu(\mathcal{I}_n;\gamma,t) = v(t,S,y),$$

where the pricing function v solves the quasilinear equation

$$v_t + \tilde{\mathcal{L}}^{(S,y)}v - \frac{1}{2}\gamma(1-\rho^2)a^2v_y^2 = \sum_{i=1}^{n}(v \wedge K_i)\cdot\delta(t-t_i), \tag{5.23}$$

for $(t,S,y) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}$, with terminal condition $v(T,S,y) = g(S,y)$, where $v \wedge K_i = \min\{v, K_i\}$ and δ is the Dirac function.[4] *Moreover, the corresponding optimal contract terminating time is given by $\tau^*(\omega) = \min\{t_i : v(t_i, S_{t_i}, Y_{t_i})(\omega) \leq K_i(S_{t_i}, Y_{t_i})(\omega)\}$.*

In what follows, we discuss some properties of the indifference price. It is worth noting that the pricing PDE (5.23) can be characterized by its absence of the drift in S and the adjustment of the drift in y. The absence of the drift in S is by no mean an accidental observation. Since the stock S is tradable, the ingredient of the derivative risk generated by the stock S can be fully hedged through trading in S. So by no-arbitrage argument the drift in S should be exactly the interest rate, which is zero by assumption. This observation is well known in the Black-Scholes framework. On the other hand, the adjustment of the drift in y reflects the correlation of S and Y. Although the risk factor Y is nontraded, the investor can hedge partially the risk component associated with Y by trading in S that is correlated to Y. As a consequence, the market price of risk related to Y is reduced by the amount of $\rho\frac{\mu}{\sigma}$, which leads to the drift term $\left(\frac{b}{a} - \rho\frac{\mu}{\sigma}\right)av_y$. In fact, as $\rho \to 1$, the market becomes complete because of the perfect correlation between S and Y. So the market prices of the two risky assets must be consistent with each other, namely $\frac{\mu}{\sigma} = \frac{b}{a}$, to exclude arbitrage opportunity, and therefore the drift in y would vanish as well. In this case, we further observe that the pricing PDE (5.23) becomes linear and is independent of γ.

Another notable feature of the equation (5.23) is that the nonlinear term is of quadratic form and the only term involving the risk aversion parameter γ. By using the comparison theorem, one can show that the buyer's indifference price

[4] The Dirac function δ is introduced for notation convenience. It formally represents the pasting condition $v(t_i^-, S, y) = (v(t_i,S,y) - K_i(S,y))^+$ at time t_i.

as a function of γ is non-increasing. Thus the indifference price is bounded by $\nu(\mathcal{I}_n;\gamma,t) \leq \nu(\mathcal{I}_n;0,t)$, where $\nu(\mathcal{I}_n;0,t)$ is the indifference price as $\gamma \to 0$.

Corollary 5.3.6. *As $\gamma \to 0$, the (risk neutral) indifference price, $\nu(\mathcal{I}_n;0,t) = \phi^n(t,S,y)$, solves the linear equation*

$$\phi_t + \tilde{\mathcal{L}}^{(S,y)}\phi = \sum_{i=1}^{n}(\phi \wedge K_i) \cdot \delta(t-t_i), \tag{5.24}$$

with terminal condition $\phi(T,S,y) = g(S,y)$. Moreover,

$$\phi^i(t,S,y) = E_{\mathbb{Q}}\left[\left(\phi^{i-1}(t,S_{t_i},Y_{t_i}) - K_i(S_{t_i},Y_{t_i})\right)^+ \big| S_t = S, Y_t = y\right], \tag{5.25}$$

for $(t,S,y) \in [0,t_i] \times \mathbb{R}^+ \times \mathbb{R}$, where $\phi^0 = E_{\mathbb{Q}}\left[g(S_T,Y_T)|S_t = S, Y_t = y\right]$ and the martingale measure $\mathbb{Q}$ is defined in (5.15).

In other words, as the buyer becomes risk neutral, the indifference price converges to the expectation of the payoff under the minimal entropy measure. This asymptotic behavior has been well studied under dual arguments by various authors; see, among others, [118] and [40]. In the current setting, the equation (5.24) follows from the robustness of viscosity solutions [see 89, Proposition I.3] for equation (5.23) and the uniqueness of solution to the linear equation (5.24). The probabilistic representation (5.25) is a direct result of the Feynman-Kač formula and Girsanov theorem.

5.4 The Risk Monitoring Strategy

In this section, we follow the spirit of [111] to construct the risk monitoring strategy for an arbitrary installment option $\mathcal{I}_n(S,Y)$.

We first recall that the optimal control policy for the investment problem (5.7) is provided in the feedback form

$$\pi^*(t,S,x,y) = \frac{1}{\gamma}\frac{\mu(t,S)}{\sigma^2(t,S)} - Sm_S(t,S) - Sv_S(t,S,y) - \rho\frac{a(t,S)}{\sigma(t,S)}v_y(t,S,y),$$

for $t \in [t_1, T]$. Therefore, the optimal control process $\Pi_s^{n,*}$ for $s \in [t, T]$ is given by

$$\Pi_s^{n,*} = \frac{1}{\gamma}\frac{\mu(s, S_s)}{\sigma^2(s, S_s)} - S_s m_S(s, S_s) - S_s v_S(s, S_s, Y_s) - \rho\frac{a(s, S_s)}{\sigma(s, S_s)}v_y(s, S_s, Y_s),$$

with its optimality following from the classical verification results [see, for example, 50, Theorem IV.3.1]. Recalling (5.6), we obtain the optimal wealth process

$$X_s^* = x - v(t, S_t, Y_t) - \sum_{i=1}^{n} K_i(S_{t_i}, Y_{t_i})\mathbf{1}_{\{t<t_i<s\wedge\tau^*\}} + \int_t^s \mu\Pi_u^* du + \int_t^s \sigma\Pi_u^* dW_u^1, \quad (5.26)$$

for $s \in [t, T]$, where μ and σ depend on (u, S_u).

Respectively, the optimal trading policy of the Merton problem (5.5), $\Pi_s^{0,*}$, is given by $\Pi_s^{0,*} = \pi^{0,*}(s, S_s, X_s^{0,*})$ for $s \in [t, T]$, where the feedback function $\pi^{0,*}$ is

$$\pi^{0,*}(t, S, x) = \frac{1}{\gamma}\frac{\mu(t, S)}{\sigma^2(t, S)} - S m_S(t, S),$$

for $(t, S, x) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}$. The optimal wealth process is then given by

$$X_s^{0,*} = x + \int_t^s \mu(u, S_u)\Pi_u^{0,*} du + \int_t^s \sigma(u, S_u)\Pi_u^{0,*} dW_u^1, \quad (5.27)$$

for $s \in [t, T]$.

Imitating the arguments used for "delta hedging" in Black-Scholes setting, we consider the optimal "indifference hedging strategy," which is defined as the adjustment of optimal portfolio caused by incorporating the derivative. This leads to the following.

Proposition 5.4.1. *The hedging strategy for the installment option $\mathcal{I}_n(S, Y)$ is given by*

$$\Delta_s^n := \frac{\Pi_s^{n,*} - \Pi_s^{0,*}}{S_s} = -v_S(s, S_s, Y_s) - \rho\frac{a(s, S_s)}{\sigma(s, S_s)S_s}v_y(s, S_s, Y_s), \quad (5.28)$$

for $s \in [t, T]$, where the pricing function v is given in (5.23).

The above hedging strategy Δ_s^n represents the optimal number of shares of the traded asset that the investor have in his/her portfolio due to the presence of the installment option $\mathcal{I}_n$. For more discussions, see, for example, [111, 107], and [132]. The following result follows directly from equation (5.23) and Itô calculus.

Proposition 5.4.2. *The indifference price process, $\nu_s = v(s, S_s, Y_s), t \leq s \leq T$, satisfies*

$$\begin{aligned}\nu_s = v(t, S_t, Y_t) + \sum_{i=1}^{n} K_i \mathbf{1}_{\{t<t_i<s\wedge\tau^*\}} + \int_t^s \left(\frac{1}{2}\gamma\bar{\rho}^2 a^2 v_y^2 + \rho\frac{\mu}{\sigma} a v_y + \mu S_u v_S\right) du \\ + \int_t^s \left(\sigma S_u v_S + \rho a v_y\right) dW_u^1 + \bar{\rho}\int_t^s a v_y dW_u^{\perp}.\end{aligned} \tag{5.29}$$

We next define the residual optimal wealth process and the residual risk process; see [111] and [136].

Definition 5.4.3. *Let X_s^*, $X_s^{0,*}$ and H_s be given, respectively, by (5.26), (5.27) and (5.29). Then, we define the residual optimal wealth process by $L_s := X_s^* - X_s^{0,*}$, $s \in [t, T]$, with initial data $L_t = -v(t, S, y)$, and the residual risk process by $R_s := L_s - \nu_s$, $s \in [t, T]$, with initial data $R_t = 0$.*

From (5.26) and (5.27) we deduce

$$\begin{aligned}L_s &= -v(t, S_t, Y_t) - \sum_{i=1}^{n} K_i \mathbf{1}_{\{t<t_i<s\wedge\tau^*\}} + \int_t^s \mu(\Pi_u^* - \Pi_u^{0,*})du + \int_t^s \sigma(\Pi_u^* - \Pi_u^{0,*})dW_u^1 \\ &= -v(t, S_t, Y_t) - \sum_{i=1}^{n} K_i \mathbf{1}_{\{t<t_i<s\wedge\tau^*\}} - \int_t^s \left(S_u v_S + \rho\frac{a}{\sigma}v_y\right)\left(\mu du + \sigma dW_u^1\right),\end{aligned} \tag{5.30}$$

where μ, σ depend on (u, S_u), a on (u, Y_u) and v on (u, S_u, Y_u). Further, combining (5.29) and (5.30) yields

$$R_s := L_s + \nu_s = \frac{1}{2}\gamma\bar{\rho}^2\int_t^s a^2 v_y^2 du + \bar{\rho}\int_t^s a v_y dW_u^{\perp}. \tag{5.31}$$

The following result provides the payoff decomposition.

Theorem 5.4.4. *The payoff $g(S_T, Y_T)$ of the installment option $\mathcal{I}_n(S, Y)$ with n*

outstanding installments $K_n, \cdots, K_1$ admits the decomposition

$$
\begin{aligned}
g(S_T, Y_T) = v(t, S_t, Y_t) &+ \sum_{i=1}^{n} K_i(S_{t_i}, Y_{t_i}) \mathbf{1}_{\{t<t_i<\tau^*\}} \\
&+ \int_t^T v_S(s, S_s, Y_s) dS_s + \int_t^T \rho \frac{a}{\sigma} v_y(s, S_s, Y_s) \frac{dS_s}{S_s} \\
&+ \frac{1}{2}\gamma\bar{\rho}^2 \int_t^T a^2 v_y^2(s, S_s, Y_s) ds + \bar{\rho} \int_t^T a v_y(s, S_s, Y_s) dW_s^{\perp},
\end{aligned} \tag{5.32}
$$

where $v(t, S, y)$ is the pricing function given by (5.23), and $\tau^*(\omega) = \min\{t_i : v(t_i, S_{t_i}, Y_{t_i}) \leq K_i(S_{t_i}, Y_{t_i})\}$ is the contract's optimal terminating time.

Proof. It follows from (5.30) that the residual wealth at terminal time T is given by

$$
L_T = -v(t, S_t, Y_t) - \sum_{i=1}^{n} K_i(S_{t_i}, Y_{t_i}) \mathbf{1}_{\{t<t_i<\tau^*\}} - \int_t^T \left(S_s v_S + \rho \frac{a}{\sigma} v_y \right) \frac{dS_s}{S_s}.
$$

By the residual risk process (5.31), we have

$$
L_T + \nu_T = \frac{1}{2}\gamma\bar{\rho}^2 \int_t^T a^2 v_y^2 ds + \bar{\rho} \int_t^T a v_y dW_s^{\perp}.
$$

Thus, the decomposition (5.32) follows from the last two equations since $\nu_T = g(S_T, Y_T)$. $\square$

Intuitively, the payoff of the installment option is decomposed into four components: the indifference price (upfront value), the installment payments up to τ^*, the hedgeable risk, and the residual risk. The hedgeable risk, which is characterized by the third and the fourth terms on the right-hand side of (5.32), is captured by the proceeds from the trading in S. Specifically, the integrand of the third term reflects the hedge of the risk arising from the stock S, which resembles to the delta hedge of the Black-Scholes model. The integrand of the fourth term represents the amount that is invested into the stock S in order to hedge the risk ingredient generated by the non-traded risk factor Y. This term disappears when $\rho = 0$. On the other hand, when $\rho \to 1$ and $\frac{b}{a} = \frac{\mu}{\sigma}$, the integrand of the fourth term becomes the familiar delta hedge of the Black-Scholes model. Finally, the last two terms in (5.32) represent the residual risk that is accumulated over time and cannot be further diversified.

5.5 Installment Options on Non-traded Risk Factors

In this section, we restrict our attention to an installment option $\mathcal{I}_n(Y)$ contingent only on the non-traded risk factor Y. That is, the payoff $G_T = g(Y_T)$ and installments $K_i = K_i(Y_{t_i})$ depend only on Y, where $g(\cdot)$ and $K_i(\cdot), (i = 1, \cdots, n)$ are bounded functions. In addition, the stock dynamics (5.1) is assumed to be a homogeneous SDE, namely, the coefficients $\mu(t, S) = \mu(t)$ and $\sigma(t, S) = \sigma(t)$. The motivation for this consideration is to provide a more tractable model that is rich enough for the valuation of venture capital and other sequential investments.

In this special case, the first optimization problem reduces to the classical Merton model, where the value function M is given by

$$M(t, x) = -\exp\left\{-\gamma x - \frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)} ds\right\},$$

for $(t, x) \in [0, T] \times \mathbb{R}$ [see 96]. The second optimization problem is defined as

$$V^n(t, x, y) := \sup_{\tau\in\mathcal{T},\pi\in\mathcal{A}_{[t,T]}} E\left[u\left(X_T + g(Y_T)\mathbf{1}_{\{\tau=T\}}\right)\middle| X_t = x, Y_t = y\right],$$

for $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, where (X_s) is the wealth process given by (5.6). We recall that $\tau \in \mathcal{T} := \{t_n, \cdots, t_1, t_0 = T\}$ is the contract's terminating time and $\mathcal{A}_{[t,T]}$ the set of admissible policies. Moreover, the indifference condition (5.8) becomes

$$M(t, x) = V^n(t, x - \nu(\mathcal{I}_n; \gamma, t), y),$$

for $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. The pricing equation (5.23) reduces to

$$v_t + \frac{1}{2}a^2 v_{yy} + \left(b - \rho\frac{\mu}{\sigma}a\right) v_y - \frac{1}{2}\gamma\bar{\rho}^2 a^2 v_y^2 = \sum_{i=1}^{n}(v \wedge K_i) \cdot \delta(t - t_i), \tag{5.33}$$

for $(t, y) \in [0, T] \times \mathbb{R}$, with terminal condition $v(T, y) = g(y)$. Following the similar arguments used in Theorem 2 and 3 in [111], we derive the probabilistic representation of v, and obtain the following recursive formula for the price.

Theorem 5.5.1. *Let $v^0(t, y)$ denote the pricing function for the European payoff*

$c(Y_T)$, which is given by

$$v^0(t,y) = -\frac{1}{\gamma\bar{\rho}^2}\log E_{\mathbb{Q}}\left[e^{-\gamma\bar{\rho}^2 g(Y_T)}\Big|Y_t = y\right], \tag{5.34}$$

for $(t,y) \in [0,T] \times \mathbb{R}$, with $\mathbb{Q}$ introduced in (5.15). The indifference price for the installment option $\mathcal{I}_n(Y)$ is given by

$$\nu(\mathcal{I}_n;\gamma,t) = v^n(t,y),$$

where the pricing function $v^n : [0,t_n] \times \mathbb{R} \to \mathbb{R}$ is defined recursively by

$$v^i(t,y) = -\frac{1}{\gamma\bar{\rho}^2}\log E_{\mathbb{Q}}\left[e^{-\gamma\bar{\rho}^2 (v^{i-1}(t_i,Y_{t_i})-K_i)^+}\Big|Y_t = y\right], \tag{5.35}$$

for $(t,y) \in [0,t_i] \times \mathbb{R}, (i = 1,\cdots,n)$. Moreover, the corresponding optimal contract terminating time is $\tau^*(\omega) = \min\{t_i : v^i(t_i, Y_{t_i}(\omega)) \le K_i(Y_{t_i}(\omega))\}$.

From (5.35) it follows that an installment option with only one installment reduces to a compound call on a call, which was previously considered in [61, 62, 119], and [127] in an arbitrage-free setting. In general, a compound call (resp. put) option gives the holder the right but not obligation to buy (resp. sell) a new option. The compound option has strike price K_1 and time to maturity t_1, while the underlying one has payoff $g(Y_T)$ and time to maturity $T > t_1$. Following the same line of arguments used in Theorems 5.3.5 and 5.5.1, we deduce that the indifference prices of a compound call, $C^c(t,y)$, and a compound put, $C^p(t,y)$, are given, respectively, by

$$C^c(t,Y_t) = \mathcal{E}_t\left[\left(\mathcal{E}_{t_1}[g(Y_T)] - K_1\right)^+\right],$$

and

$$C^p(t,Y_t) = \mathcal{E}_t\left[\left(K_1 - \mathcal{E}_{t_1}[g(Y_T)]\right)^+\right],$$

where $\mathcal{E}_t$ denotes the nonlinear pricing functional defined by

$$\mathcal{E}_t[G] := -\frac{1}{\gamma\bar{\rho}^2}\log E_{\mathbb{Q}}\left[e^{-\gamma\bar{\rho}^2 G}\big|Y_t\right].$$

5.6 Applications to ASX Installments and Venture Capital

In this section, we illustrate an application of the installment model to analyze the ASX installment warrants and the staged financing of venture capital. As mentioned earlier, the former is an innovative financial product traded actively on the Australian Stock Exchange (ASX), and the latter is a widely used financing tool in venture capital investments.

With these applications in mind, we shall develop a numerical procedure for pricing installment options contingent on the asset Y, such as an installment call and put with constant exercise price K. The dynamics of Y is taken to be lognormal, namely, $a(t,y) = \alpha y$ and $b(t,y) = \beta y$ with constants α and β. Unless otherwise specified, the model parameters are set in Table 5.1. These installment options have, in general, 0 to 8 equal installments of amount $K_i = 3$ or 6, where the case of 0 installment reduces to the European one.

Table 5.1: Model parameters.

Y_0	K	T	r	μ	σ	β	α	ρ	γ
100	100	1	5%	10%	25%	15%	30%	80%	0.01

In order to apply the results, we first normalize the price processes using the bond as numeraire. This yields the following discounted parameters: $\tilde{r} = 0$, $\tilde{\mu} = 5\%$, $\tilde{\beta} = 10\%$, $\tilde{K} = 100e^{-rT}$, and $\tilde{K}_i = 100e^{-rt_i}$, while those of σ, α, ρ, and Y_0 remain unchanged. The indifference price of such installment options is given by the recursive formula (5.35) with discounted installments $\tilde{K}_i$. Recalling (5.17), the dynamics of Y (in units of bond) under the measure $\mathbb{Q}$ is given by

$$dY_s = \left(\tilde{\beta} - \rho\frac{\tilde{\mu}}{\sigma}\alpha\right) Y_s ds + \alpha Y_s d\tilde{W}_s, \tag{5.36}$$

with $\tilde{W}$ being a $\mathbb{Q}$-Brownian motion. The problem amounts to computing the conditional expectation appearing in (5.35). An efficient procedure is the simple least-squares Monte Carlo (LSM) simulation, which was originally proposed by [92] and has been widely used to price American options. In the sequel, we developed a LSM procedure for the indifference price of installment options.

The key idea of LSM is based on the least-squares approximation, which, for completeness, is briefly reviewed below. We assume that the conditional expectation is an element of the space of square-integrable functions. This rather mild assumption enables us to represent it as a linear combination of a countable set of simple basis functions, i.e.

$$F(Y_{t_i};t_i) := E_{\mathbb{Q}}\left[e^{-\gamma\bar{\rho}^2 F(Y_{t_{i-1}};t_{i-1})}\middle|Y_{t_i}\right] = \sum_{k=0}^{\infty} p_k \psi_k(Y_{t_i}),$$

with constant coefficients p_k and orthonormal basis ψ_k. The LSM entails projecting the conditional expectation onto the subspace spanned by the first Ψ basis functions:

$$F(Y_{t_i};t_i) = \sum_{k=0}^{\Psi} p_k \psi_k(Y_{t_i}) + \epsilon.$$

In general, the choice of basis functions includes Chebyshev, Hermite, Jacobi, and Legendre polynomials, though numerical experiments indicate that the simple polynomials also give accurate results. Once we specify the basis functions, the problem reduces to find the best coefficients to minimize the squared error of approximation ϵ^2, which can be done using a simple regression procedure. We first generate a certain number of sample paths for the price process Y. Starting from the last period, we, in turn, compute the realization of the condition expectation F and the powers of Y, for each generated path j. Using this simulated data, we regress the conditional expectation against the powers. For instance, with a polynomials of order 2, we have the following regression model:

$$F^j = \exp\left\{-\gamma\bar{\rho}^2 F(Y^j_{t_{i-1}};t_{i-1})\right\} = p_0 + p_1 Y^j_{t_i} + p_2 (Y^j_{t_i})^2 + \epsilon^j.$$

The regression coefficients are obtained by solving a linear system. With the approximation of conditional expectation, it is easy to compute the installment value and the probability of termination at each installment date t_i. This procedure is then carried backward until the initial time.

We examine the accuracy of the LSM procedure with basis functions being polynomials, using installment calls with strike price K and 0 to 6 equal installments of amount $K_i = 3$. The numerical results are shown in Table 5.2. In the simulation, the regression procedure is based on 10,000 (5,000 plus 5,000 antithetic) sample paths of the underlying asset. The standard errors and the 90% confidence intervals are computed using 100 replications. The center of the latter thus provides an estimate of the installment value. As can be seen, the cases of order 3 and 4 provide a fairly good approximation with stable standard errors.

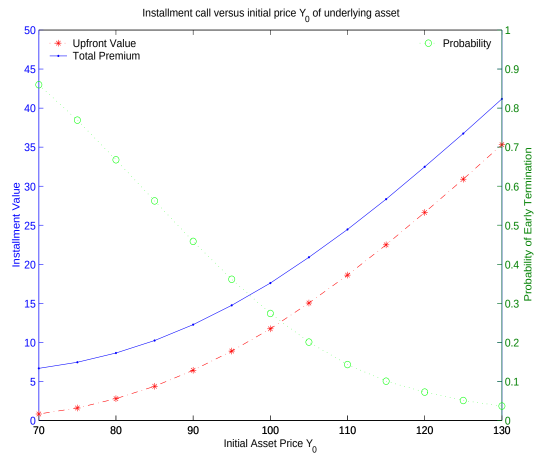
Table 5.2: Installment upfront values and order of basis functions for LSM.

	Order 2		Order 3		Order 4	
n	Value (s.e.)	90% C.I.	Value (s.e.)	90% C.I.	Value (s.e.)	90% C.I.
0	15.840 (.016)	[15.812, 15.867]	15.861 (.015)	[15.837, 15.886]	15.850 (.016)	[15.823, 15.877]
1	13.398 (.016)	[13.372, 13.424]	13.220 (.017)	[13.193, 13.248]	13.242 (.015)	[13.217, 13.267]
2	11.011 (.022)	[10.974, 11.048]	10.828 (.016)	[10.802, 10.855]	10.879 (.018)	[10.850, 10.909]
3	8.635 (.040)	[8.569, 8.701]	8.593 (.018)	[8.564, 8.622]	8.645 (.018)	[8.616, 8.675]
4	6.366 (.057)	[6.272, 6.461]	6.537 (.017)	[6.509, 6.565]	6.625 (.018)	[6.594, 6.655]
5	4.325 (.086)	[4.183, 4.467]	4.756 (.018)	[4.726, 4.786]	4.815 (.017)	[4.786, 4.843]
6	2.410 (.150)	[2.161, 2.659]	3.231 (.016)	[3.205, 3.257]	3.252 (.015)	[3.227, 3.277]

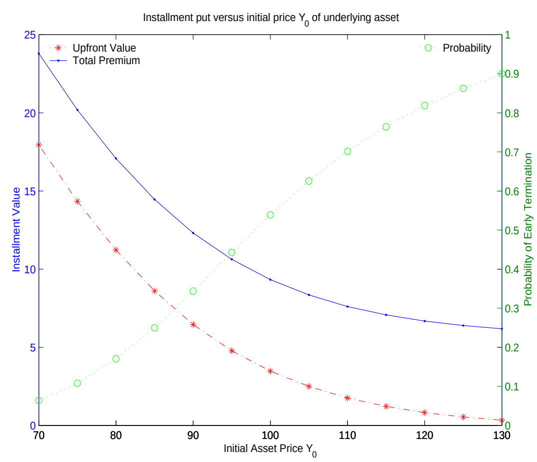
This table shows the installment upfront values, estimated by LSM simulation. The installment option is a call with equal installments ($K_i = 3$) and other parameters given in Table 5.1. The simulation for the regression procedure is based on 10,000 (5,000 plus 5,000 antithetic) sample paths from the underlying price process. The standard errors (s.e.) of the simulation estimates are given in parentheses and the 90% confidence intervals (C.I.) in brackets, which are computed using 100 replications.

In what follows, we apply the LSM procedure to investigate the dependence of the installment value on the market parameters and the installment structure, where we use polynomials of order 3 as basis functions. Figure 5.3 shows the dependence of installment values and the probability of early termination on the initial price Y_0. The graph on the left is an installment call and that on the right is a put, both of which have only one installment ($K_1 = 6$) at $T/2$. The other parameters are listed in Table 5.1. Note that the installment values displayed in the figure include the upfront value (indifference price) plotted using dash-dot line with asterisk markers and the total premium in solid line with point markers. The latter is defined as the present value of the installments plus the upfront value, namely, $v^n + \sum_{i=1}^{n} e^{-rt_i} K_i$. The probability of early termination is plotted using dotted line with circle markers.

Figure 5.4 shows the dependence of installment values and the probability of early termination on the level of installment payment. The graph on the left presents



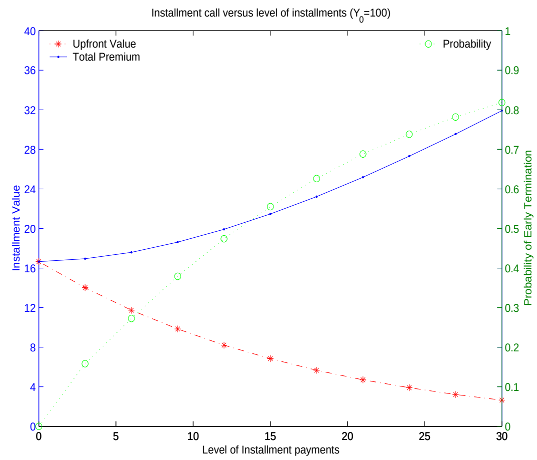
(a) Installment call



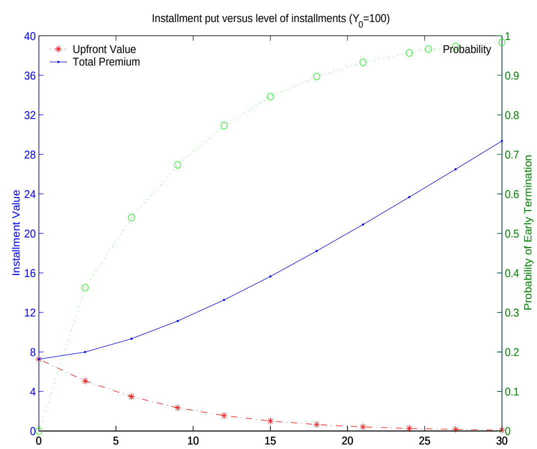
(b) Installment put

Figure 5.3: Installment call and put as functions of initial price Y_0

an installment call and that on the right is a put. Both of them have one installment K_1 of amount varying from 0 to 30 and other parameters listed in Table 5.1. As we can see, in both cases, the upfront value is monotonically decreasing with respect to the level of installments, while the total premium and the probability of early termination is monotonically increasing.



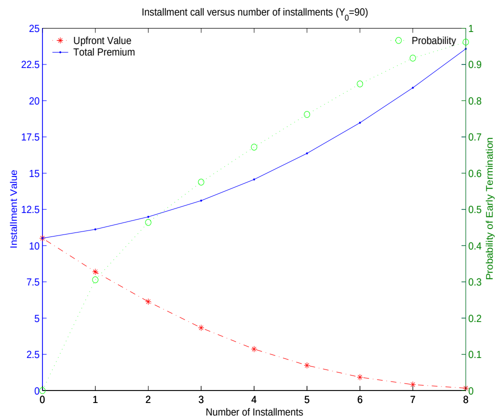
(a) Installment call



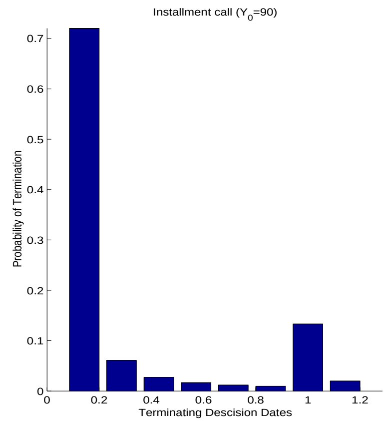
(b) Installment put

Figure 5.4: Instalment call and put as functions of the level of installment

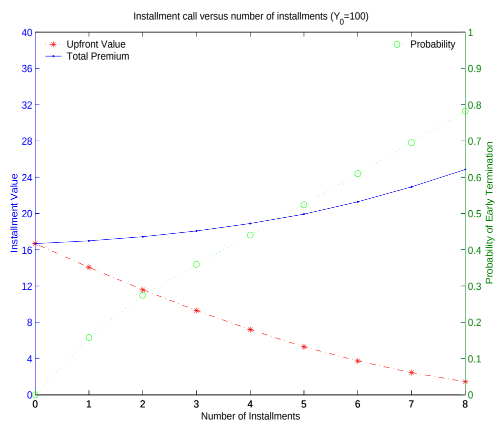
In Figure 5.5, we focus on the effect of the number of installments. The installment option in consideration is a call with equal installment ($K_i = 3$) and other parameters given in Table 5.1. The three graphs on the left show the dependence



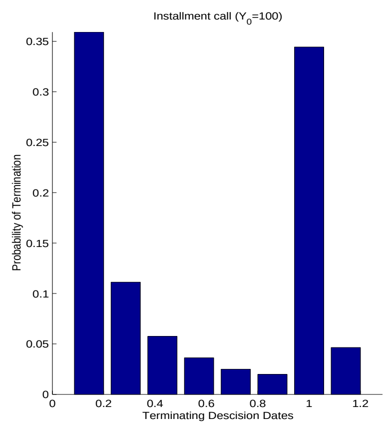
(a) Installment values ($Y_0 = 90$)



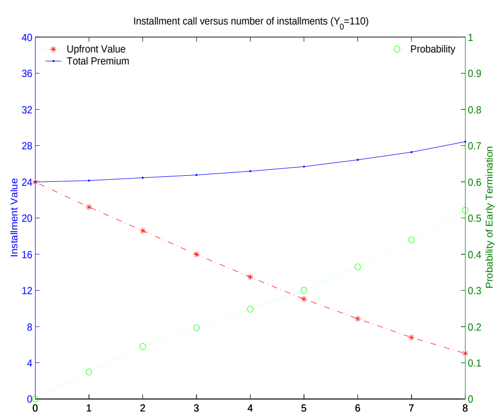
(b) Terminating decisions ($Y_0 = 90$)



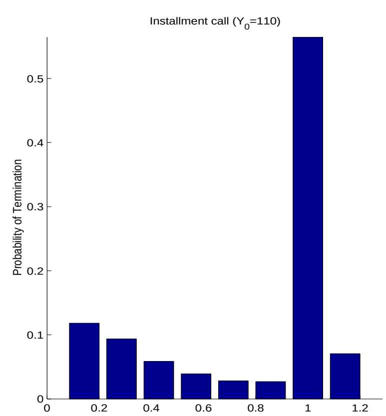
(c) Installment values ($Y_0 = 100$)



(d) Terminating decisions ($Y_0 = 100$)



(e) Installment values ($Y_0 = 110$)



(f) Terminating decisions ($Y_0 = 110$)

Figure 5.5: Dependence of installment values on the number of installments

of the installment values on the number of installments n, where Figure 5.5a, 5.5c, and 5.5e correspond, respectively, to the case of out-of-the-money ($Y_0 = 90$), at-the-money ($Y_0 = 100$), and in-the-money ($Y_0 = 110$). We observe that the dependence of installment values on the number of installments is similar to that on the level of installments as shown in Figure 5.4. Correspondingly, we investigate the distribution of contract termination decisions for the case with 6 installments, as shown in Figure 5.5b, 5.5d, and 5.5f, respectively. Note that the probability of termination observed at and after maturity $T = 1$ represents, respectively, the probability of option exercise at maturity and the probability that the option is expired without exercise.

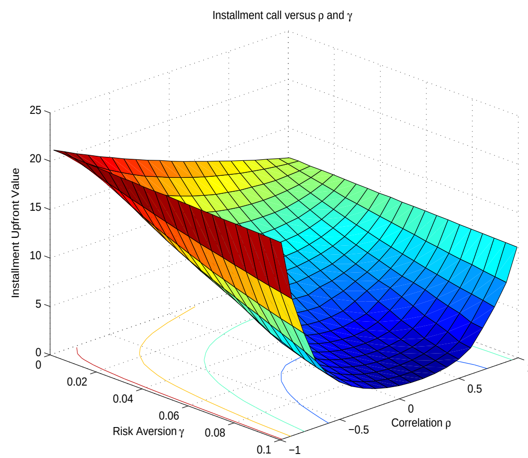
We next examine the dependence of the installment value and the probability of early termination on the risk aversion parameter γ and the correlation coefficient ρ. The installment option is a call with one installment ($K_1 = 6$) at $T/2$, and the other parameters listed in Table 5.1. As shown in Figure 5.6, the graph on the left displays the installment upfront value and that on the right presents the probability of early termination. Clearly, we observe that the indifference price (upfront value) is monotonically increasing in the risk aversion parameter γ, constant in γ at $\rho = \pm 1$, and linearly decreasing in ρ when $\gamma \to 0$. On the other hand, the probability of early termination is monotone decreasing in γ, constant in γ at $\rho = \pm 1$, and linearly increasing in ρ when $\gamma \to 0$.

5.6.1 ASX Installment Warrants

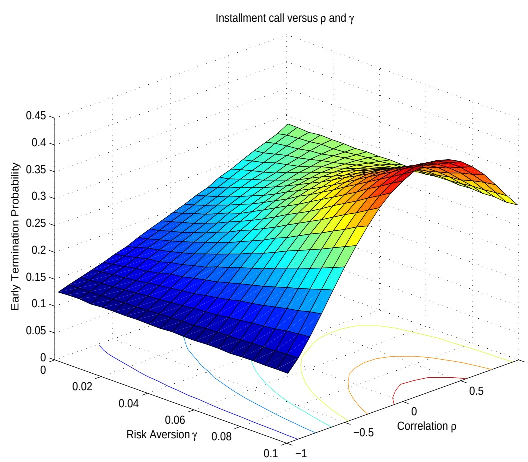
Unlike options, warrants have dilution effect on the shares of the firm, which makes it difficult to price a warrant as an ordinary option on stock. [14] suggested to price warrants as options on the firm's equity value. In the current setting, one can view the equity as the process Y. Following the analysis of [88], an installment warrant can be taken as an installment option with terminal payoff

$$g(Y_T) = \frac{cN}{N + cM} \left(\frac{Y_T}{N} - K \right)^+,$$

where M, N, and c denote the number of outstanding warrants, the number of shares, and the conversion ratio, respectively. In what follows, we apply the installment model and the LSM procedure, developed previously, to analyze this product.



(a) Installment upfront value



(b) Probability of early termination

Figure 5.6: Installment call as a function of risk aversion γ and correlation ρ

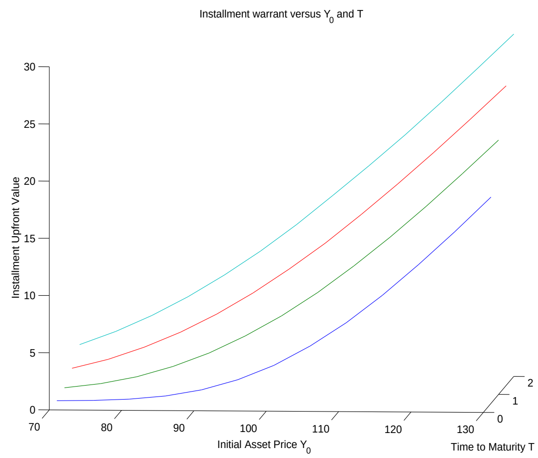
Table 5.3 shows the upfront value and the total premium of installment warrants for various degrees of dilution and number of installments. The installment warrants have equal installments ($K_i = 3$) and the other model parameters are listed in Table 5.1. Without loss of generality, we normalize the number of shares to one, and we also take the conversion ratio $c = 1$. From Table 5.3, we observe that the total premium is increasing in the number of installments, while the upfront values of installment warrants is decreasing. The reason for the former is that the presence of installments provides the option of early termination, which demand extra premium. However, as we increase the total discounted installment payments, the upfront payment is expected to decrease. We further observe that both the upfront value and the total premium decrease significantly with the number of outstanding warrants. This is due to the dilution effect of warrants. The more the outstanding warrants, the less is the value of the shares, and so is the warrants. Such dilution effect further weakens as the number of installments increases. As a result, the installment warrants have a weaker dilution effect than the European counterpart and the design of the installment structure provides more flexibility to control the firm's capital dilution.

Table 5.3: Installment warrant values and dilution effect.

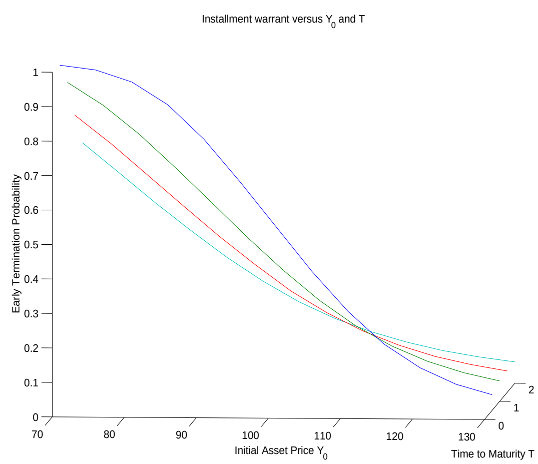
	$M = 0.0$		$M = 0.1$		$M = 0.5$		$M = 1.0$		$M = 2.0$	
n	Upfront	Total	Upfront	Total	Upfront	Total	Upfront	Total	Upfront	Total
0	16.652 (.017)	16.652	15.231 (.016)	15.231	11.317 (.012)	11.317	8.581 (.010)	8.581	5.778 (.006)	5.778
1	14.026 (.018)	16.952	12.631 (.016)	15.557	8.783 (.012)	11.709	6.124 (.009)	9.050	3.478 (.007)	6.404
2	11.563 (.016)	17.416	10.220 (.016)	16.072	6.490 (.011)	12.342	4.016 (.009)	9.868	1.740 (.006)	7.592
3	9.289 (.016)	18.067	7.966 (.015)	16.744	4.485 (.012)	13.263	2.319 (.009)	11.097	0.650 (.004)	9.428
4	7.183 (.019)	18.888	5.943 (.015)	17.648	2.841 (.011)	14.546	1.141 (.007)	12.845	0.168 (.002)	11.872

This table shows the upfront value and total premium of installment warrants with equal installments ($K_i = 3$). The simulation for the regression procedure is based on 10,000 (5,000 plus 5,000 antithetic) sample paths from the underlying price process. The standard errors given in parentheses are computed using 100 replications and the reported installment values are their sample mean.

Figure 5.7 presents the dependence of the upfront value and the probability of early termination on the initial equity value Y_0 and the time to maturity T. The installment warrant has one installment ($K_1 = 6$) at $T/2$ and the number of outstanding of warrants is $M = 0.5$.



(a) Installment upfront value



(b) Probability of early termination

Figure 5.7: Installment warrants versus initial equity value Y_0 and maturity T

5.6.2 Staged Financing of Venture Capital

In the sequel, we illustrate the application to analyze the staged financing of venture capital. As discussed previously, the staged financing is a widely used tool in venture investments such as high-tech startups. These innovative startup companies are commonly characterized by their high risk feature and low rate of success. Indeed, many venture projects may face many years of negative earnings before seeing any profits. According to [13], the fraction of such projects that investors can successfully cash out – mostly through IPO's – is less than twenty percent. To keep their ventures under control, venture capitalists typically finance the committed funds in stages rather than providing the total amount of capital upfront. The financing in the subsequent stages is contingent on the performance of the project development. If at any stage it fails to meet the expected target, the venture capitalist has the option to abandon the venture and liquidate the project. In the real options context, such a staging covenant can be viewed as an installment option; see, for example, [38] and the references therein.

As an example, we consider a venture project with state variable (cashflow value) modelled as the non-traded asset Y, which is taken to be lognormal as before. Its staging parameters are listed in Table 5.4. These parameters are taken from the empirical data reported in [63, Table V]. The various financing stages are grouped into either startup, early rounds, middle rounds, or final rounds. Similar to the classification of [63], we classify the 'seed' and 'startup' investments as startup stage. The 'early stage' and 'first stage' are named as early rounds, while 'other early' and 'expansion' investments are considered as middle rounds, Finally, the 'second', 'third', and 'bridge' stages are classified as final stage financing. The amount of funding for each classified round in Table 5.4 is defined as the total funding of the included stages. The market parameters are given as follows: $r = 5\%$, $\mu = 10\%$, $\sigma = 20\%$, $\beta = -5\%$, and $\alpha = 60\%$. The negative value of return β and the high level of volatility α are chosen to reflect the high-risk feature of venture investments.

Table 5.4: Staging parameters of venture capital

Type of Financing	Startup	Early Round	Middle Round	Final Round
Duration of Stage[5] (years)	1.21	1.08	1.08	1.01
Amount of Funding ($ million)	2.387	2.982	4.525	7.634

From an investor's perspective, such staged venture project is essentially an installment call with strike price $K = 7.634$ and the three installment payments

[5] Duration of stage is defined as time to the next funding.

listed in Table 5.4. Based on this observation, the installment model provides a framework for analyzing the staged financing of venture investments. For instance, Table 5.5 summarizes its net values for various risk aversion parameters and correlation coefficients, where the initial state variable $Y_0 = 50$. As it can be seen, the net values of the venture project decrease significantly with the investor's risk aversion. The reason lies in the fact that venture projects are high risk investments. A more risk-averse investor would demand higher risk premium for bearing the ventures. Consequently, an identical venture contract is less valuable for an investor with higher risk aversion.

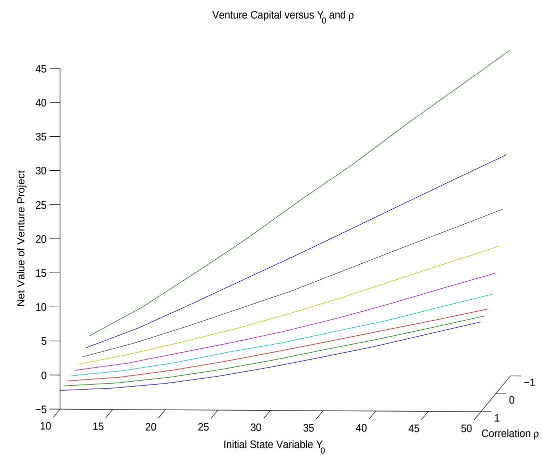
The effect of the correlation coefficient ρ is ambiguous. The increase of ρ, on one hand, depresses the drift of Y under the pricing measure $\mathbb{Q}$ (cf. (5.36)), which decreases the future cashflows and makes the project less valuable. On the other hand, it reduces the effective risk aversion $\gamma\bar{\rho}$ and lowers the investor's demand of risk premium, thereby increases the value of the project. When γ is sufficiently small, the second effect is dominated by the first one, so that the net values tend to decrease with respect to ρ.

Table 5.5: Venture capital net value versus risk aversion γ and correlation ρ.

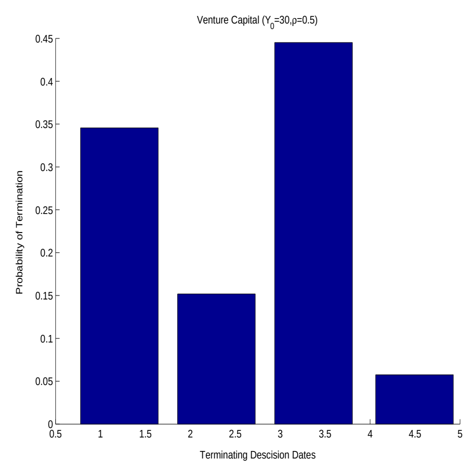
	$\gamma = 0.1$		$\gamma = 0.05$		$\gamma = 0.01$		$\gamma = 0.005$		$\gamma = 0.001$	
ρ	Value	(s.e.)	Value	(s.e.)	Value	(s.e.)	Value	(s.e.)	Value	(s.e.)
0.00	0.030	(0.003)	3.864	(0.000)	12.697	(0.017)	15.797	(0.039)	19.808	(0.048)
0.25	−0.641	(0.002)	2.758	(0.002)	10.269	(0.020)	13.102	(0.044)	16.164	(0.036)
0.50	−0.646	(0.001)	2.373	(0.005)	8.783	(0.054)	11.090	(0.036)	13.029	(0.036)
0.75	0.402	(0.003)	2.942	(0.008)	8.395	(0.038)	9.427	(0.028)	10.441	(0.038)
0.99	7.374	(0.025)	7.718	(0.032)	8.181	(0.034)	8.163	(0.034)	8.224	(0.040)

This table exhibits the net values of venture project with staging parameters given in Table 5.4. The market parameters are: $r = 5\%$, $\mu = 10\%$, $\sigma = 20\%$, $\beta = -5\%$, $\alpha = 60\%$ and $Y_0 = 50$. The simulation for the regression procedure is based on 10,000 (5,000 plus 5,000 antithetic) sample paths from the underlying price process. The standard errors (s.e.) given in parentheses are computed using 100 replications.

We next consider an investor with risk aversion parameter $\gamma = 0.01$. Figure 5.8a plots the net value of the venture project as a function of the initial state variable Y_0 and for a range of correlation coefficients ρ. It shows that ρ has a significant effect on the net value of the project. Figure 5.8b presents the probability of early termination when the initial state $Y_0 = 30$ and the correlation coefficient $\rho = 0.5$.For this particular project, the investor provides the 'startup' funding of amount \$2.387 million at time 0. With probability of about 35%, s/he would not finance the 'early' round, thereby abandoning the venture at year 1.21. Likewise, the probabilities of early termination at the 'middle' and 'final' rounds are about 15% and 45%, respectively. The probability that the investor would finance all the committed funds and finally cash out the investment turns out to be about 5%.



(a) Net value of venture capital



(b) Probability of early termination

Figure 5.8: Venture capital versus initial state Y_0 and correlation ρ

5.7 Conclusions

We have developed a utility-based approach to value a variety of installment options in incomplete markets. Examples include installment call and put options, compound options, ASX installment warrants, and staged financing of venture capital investments. For a generic installment option written on both tradable and non-traded risk factors, we have constructed the indifference price in terms of a quasilinear PDE and analyzed the associated risk monitoring strategies. In particular, we obtained an explicit nonlinear pricing formula for options contingent exclusively on the non-traded risk factor.

We further developed a Monte Carlo procedure, based on the least-squares approximation, to compute the indifference prices. Through various numerical experiments, we investigated the dependence of the indifference price and the probability of early termination on the installment structure (such as the level and the number of installments) and other market parameters (such as initial asset price, time to maturity, risk aversion parameter, and correlation coefficient). The model was further applied to analyze the dilution effect of ASX installment warrants and the staged financing of venture capital investments.

In the future, we plan to apply and extend the results developed herein to analyze other sequential investments, installment loans, and many corporate liabilities with sequential opportunities (such as equity with coupon bearing bonds, bonds with extended maturities, and callable bonds with coupon payments). Another possible direction would be the combination of static and dynamic hedging strategies for installment options.

A question of independent interest is to investigate how to apply the present approach to analyze the optimal design of installment warrants and agency problems arising in venture capital.

Chapter 6

The Valuation of Volatility Derivatives based on Dynamic Performance Criteria

6.1 Introduction

Since the stock market crash in 1987, the study of volatility has become more and more important in quantitative finance. Recently, the research has been directed to the valuation of volatility derivatives. However, the current literature is concentrated on the Black-Scholes setting and static hedging framework; see [18, 55, 58] and the reference therein.

In this chapter, we apply the utility-based approach to value a family of volatility derivatives, whose payoff, at maturity T, can be written as $G_T = g(Y_T, Z_T)$, where Y is the underlying process that the volatility depends on, and

$$Z_s = \int_t^s \zeta(\tau, Y_\tau) d\tau, \quad t \leq s \leq T. \tag{6.1}$$

For technical reasons arising in the underlying utility maximization, we assume that g and ζ are deterministic functions such that G_T is bounded and satisfy the integrability condition.

This payoff specification includes many popular volatility derivatives actively traded in the market, such as *variance swaps* and *volatility swaps*, as well as European options. In practice, variance swap is indeed a forward contract on the realized variance. Its payoff, at time T, is given by

$$N \times A \times \left\{ \frac{1}{N} \sum_{i=1}^{N} \left[\log \left(\frac{S_i}{S_{i-1}} \right) \right]^2 - \left[\frac{1}{N} \log \left(\frac{S_N}{S_0} \right) \right]^2 \right\} - N \times K_{var}, \tag{6.2}$$

where N is the notional amount of the swap, A is the annualization factor and K_{var} is the strike price.

6.2 Market and Investment Environment

We consider a dynamic financial market consisting of a riskless bond B (money-market account) and a risky stock S. The stock price is assumed to follow a diffusion process satisfying

$$dS_t = \mu(t, Y_t) S_t dt + \sigma(t, Y_t) S_t dW_t^1, \quad t \geq 0, \tag{6.3}$$

with $S_0 > 0$. The drift μ and volatility σ of the stock are driven by a stochastic factor Y, which is modelled as a correlated diffusion

$$dY_t = b(t, Y_t) dt + a(t, Y_t)(\rho dW_t^1 + \bar{\rho} dW_t^{\perp}), \quad t \geq 0, \tag{6.4}$$

with $\rho \in (-1, 1)$ being the correlation coefficient and $\bar{\rho} = \sqrt{1 - \rho^2}$. The bond is assumed to be tradable over the time horizon $[0, T]$, yielding constant interest rate r. Without loss of generality we can take $r = 0$. The results for $r > 0$ follow directly from the standard rescaling arguments by using bond as numeraire.

In equations (6.3) and (6.4), the processes $(W_t^1, W_t^{\perp}; t \geq 0)$ are independent standard Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t \geq 0}$ is generated by both W^1 and $W^{\perp}$ satisfying the usual conditions. The market coefficients μ, σ, a and b are assumed to satisfy all the regularity conditions such that equations (6.3) and (6.4) have a unique strong solution satisfying $S_t > 0$ $\mathbb{P}$-a.s. for a.e. $t \in [0, T]$. We denote by $\lambda_t = \lambda(t, Y_t), t \geq 0$, the *Sharpe ratio* process of the stock, where

$$\lambda(t, y) = \frac{\mu(t, y)}{\sigma(t, y)}. \tag{6.5}$$

We consider an investor, who trades dynamically between the bond and the stock over the time horizon $[0,T]$. Suppose that the investor starts with initial wealth $X_t = x \in \mathbb{R}$ at time t and, subsequently, uses a self-financing strategy to rebalance her portfolio allocations. At time $s \in [t,T]$, she invest the amount of π_s into the stock market (say, $\Delta_s = \pi_s/S_s$ shares), and deposits the remainder of her wealth into the bond account. It follows that the total current wealth X_s, in absence of intermediate consumption, satisfies the following controlled diffusion equation

$$dX_s = \mu(s,Y_s)\pi_s ds + \sigma(s,Y_s)\pi_s dW_s^1, \tag{6.6}$$

for $s \in [t,T]$, with $X_t = x \in \mathbb{R}$ [see 96]. The set of admissible policies is defined by $\mathcal{A} := \{\pi : \pi_s \text{ is } \mathcal{F}_s \text{ predictable and } E\int_t^T \sigma^2(s,Y_s)\pi_s^2 ds < \infty\}$.

In the current model the bond and the stock are the only liquid securities in the market, which are tradable at any time. Although the investor can buy or sell the volatility derivatives, their underlying Y_t is not tradable. This intrinsic market incompleteness and the path-dependent feature are the central issues that we will tackle in this chapter.

6.3 Valuation based on Dynamic Performance Criteria

We devote this section to review some basic concepts of dynamic performance criteria. The following definition was proposed by [113, 114].

Definition 6.3.1. *An $\mathcal{F}_t$-adapted process $U(t,x)$, $t \geq 0$, is a dynamic performance criterion if*
(i) as a function of x, it is increasing and concave for each $t \geq 0$
(ii) for each admissible self-financing strategy $\pi \in \mathcal{A}$, $U(t, X_t^\pi)$ is a supermartingale with respect to the filtration $\mathbb{F}$, i.e.

$$U(t, X_t^\pi) \geq E\big[U(s, X_s^\pi)\big|\mathcal{F}_t\big], \tag{6.7}$$

for $0 \leq t \leq s$, and
(iii) there exists some admissible self-financing strategy π^*, such that $U(t, X_t^{\pi^*})$ is a martingale with respect to $\mathbb{F}$, i.e.

$$U(t, X_t^{\pi^*}) = E\big[U(s, X_s^{\pi^*})\big|\mathcal{F}_t\big], \tag{6.8}$$

for $0 \leq t \leq s$.

It is worth noting that the dynamic performance is closely related to the investment opportunities (market environment). Property (i) makes the dynamic performance act as a traditional utility function locally. Property (ii) imposes a time consistency constraint on the investment horizons. Finally, property (iii) takes into account the investment optimality.

The first example of dynamic performance arises from the classical Merton model, in which investors aim to maximize their expected utility of terminal wealth, say, at time $\bar{T} > T$. Its value function is defined as [1]

$$M(t,x,y) := \sup_{\mathcal{A}} E_t^{x,y}\left[\bar{u}(X_{\bar{T}})\right], \tag{6.9}$$

for $(t,x,y) \in [0,\bar{T}] \times \mathbb{R} \times \mathbb{R}$, where the wealth process (X_s) satisfies (6.6) and $\mathcal{A}$ is the set of admissible policies.

It is straightforward to verify that the process defined by $V(t,x) := M(t,x,Y_t)$ satisfies the properties in definition 6.3.1. It thus defines a dynamic performance criterion, which is called the *backward dynamic performance* because it is normalized at the future time $\bar{T}$ (say, $V(T,x) = \bar{u}(x)$), and is self-generated backward in time. In particular, consider a *constant absolute risk aversion* investor with exponential utility function given by

$$\bar{u}(x) := -e^{-\gamma x}, \quad x \in \mathbb{R}, \tag{6.10}$$

where $\gamma > 0$ is the risk aversion parameter. The associated dynamic performance criteria is then given by

$$V(t,x) = -e^{-\gamma x}\left(E_{Q_0}\left[\exp\left\{-\frac{1}{2}\bar{\rho}^2\int_t^{\bar{T}} \lambda_s^2 ds\right\}\Big| Y_t\right]\right)^{\frac{1}{\bar{\rho}^2}}, \tag{6.11}$$

for $(t,x) \in [0,\bar{T}] \times \mathbb{R}$; see [129].

[1] Throughout, we adopt the short notation $E_t^{x,y,z}[\cdot] := E_{\mathbb{P}}[\cdot|X_t = x, Y_t = y, Z_t = z]$ for the conditional expectation under the historical measure $\mathbb{P}$.

6.4 Forward Indifference Valuation

In this section, we focus our attention on the indifference valuation based on dynamic performance criteria.

Following [113], we consider the dynamic performance criterion of exponential type given by

$$U(t,x) := -\exp\left\{-\gamma x + \frac{1}{2}\int_0^t \lambda_s^2 ds\right\}, \quad t \geq 0, \tag{6.12}$$

with λ being the *Sharoe ratio* process defined in (6.5). Note that such a performance is forward normalized with an exponential initial performance $U(0,x) = -e^{-\gamma x}$ at time 0. It is straightforward to check that U defined in (6.12) satisfies those three properties of Definition 6.3.1. For more detail discussion, we refer to [114].

As discussed previously, the indifference price is defined through two optimal investment problems. In absence of derivatives, the investor solves the optimization problem

$$\sup_{\mathcal{A}} E_t^{x,y,z}\big[U(t,X_T)\big], \tag{6.13}$$

where the wealth process (X_s) satisfies (6.6) and $\mathcal{A}$ is the set of admissible policies. In the second optimal investment problem, we assume that the investor buys the volatility derivative G_T at time $t \in [0,T]$. Then the relevant optimal investment problem is given by

$$\sup_{\mathcal{A}} E_t^{x,y,z}\big[U(t,X_T+G_T)\big], \tag{6.14}$$

where the wealth process (X_s) satisfies (6.6) and $\mathcal{A}$ is the set of admissible policies.

We now proceed to define the forward indifference price; see [114].

Definition 6.4.1. *The buyer's forward indifference price (BFIP) process $\nu_t(G_T)$ is defined as the amount such that*

$$\sup_{\mathcal{A}} E_t^{x,y,z}\big[U(t,X_T)\big] = \sup_{\mathcal{A}} E_t^{x-\nu_t,y,z}\big[U(t,X_T+G_T)\big], \tag{6.15}$$

for $(t,x,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$.

In the above definition, the seller's forward indifference price (SFIP) is given by $-\nu_t(-G_T)$. In order to determine the indifference price $\nu_t(G_T)$, we need to solve the two associated optimization problems (6.13) and (6.17). The first optimization problem is easy. Thanks to the martingale property of the dynamic performance $U(t, X_t^*)$, we immediately see that

$$U(t,x) = \sup_{\mathcal{A}} E_t^{x,y,z}\big[U(t,X_T)\big], \tag{6.16}$$

for $(t,x) \in [0,T] \times \mathbb{R}$.

To solve the second optimization problem, we first observe that

$$\begin{aligned} \sup_{\mathcal{A}} E_t^{x,y,z}\big[U(t,X_T+G_T)\big] &= \sup_{\mathcal{A}} E_t^{x,y,z}\left[-\exp\left\{-\gamma(X_T+G_T)+\frac{1}{2}\int_0^T \lambda_s^2 ds\right\}\right] \\ &= \exp\left\{\frac{1}{2}\int_0^t \lambda_s^2 ds\right\} u(t,x,y,z), \end{aligned} \tag{6.17}$$

where we use the following value function

$$u(t,x,y,z) = \sup_{\mathcal{A}} E_t^{x,y,z}\left[-\exp\left\{-\gamma\big[X_T+g(Y_T,Z_T)\big]+\frac{1}{2}\int_t^T \lambda^2(s,Y_s)ds\right\}\right], \tag{6.18}$$

for $(t,x,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$.

The Hamilton-Jacobi-Bellman (HJB) equation associated with problem (6.18) is given by

$$u_t + \max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 u_{xx} + \pi\left(\rho\sigma a u_{xy} + \mu u_x\right)\right) + \mathcal{L}^y u + \zeta u_z + \frac{1}{2}\lambda^2 u = 0, \tag{6.19}$$

for $(t,x,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$, with terminal condition $u(T,x,y,z) = -\exp\{-\gamma(x+g(y,z))\}$, where the differential operator $\mathcal{L}^y := \frac{1}{2}a^2\frac{\partial^2}{\partial y^2} + b\frac{\partial}{\partial y}$.

We next derive the value function u, following closely the arguments in [111].

Theorem 6.4.2. *The value function u defined in (6.18) is given by*

$$u(t,x,y,z) = -e^{-\gamma x}\left(E_{\mathbb{Q}_0}^{t,y,z}\left[e^{-\gamma\bar{\rho}^2 g(Y_T,Z_T)}\right]\right)^{\frac{1}{\bar{\rho}^2}}, \tag{6.20}$$

for $(t,x,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, where the pricing measure Q_0 is the minimal martingale measure *(MMM) defined through its density*

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \exp\left\{ -\int_0^T \lambda_s^2 dW_s^1 - \frac{1}{2}\int_0^T \lambda_s^2 ds \right\}. \tag{6.21}$$

Proof. Using the scaling property of the objective function and the structure of the controlled wealth dynamics, we postulate a solution of the form

$$u(t,x,y,z) = -e^{-\gamma x} f(t,y,z)^{\frac{1}{1-\rho^2}}. \tag{6.22}$$

Substituting into (6.19), we deduce that the function f solves equation

$$f_t + \frac{1}{2}a^2 f_{yy} + (b - \rho\lambda a) f_y + \zeta f_z = 0, \tag{6.23}$$

for $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+$, with terminal condition $f(T,y,z) = e^{-\gamma\bar{\rho}^2 g(y,z)}$.

By the Girsanov theorem, we observe that, under the minimal martingale measure $\mathbb{Q}_0$ defined in (6.21), the process $\tilde{W}_t^1 = W_t^1 + \rho\lambda t$ is a standard Brownian motion. It follows that, under $\mathbb{Q}_0$, the dynamics of Y_t and Z_t are given by

$$dY_t = \big(b(t,Y_t) - \rho\lambda(t,Y_t)a(t,Y_t)\big)dt + a(t,Y_t)\left(\rho d\tilde{W}_t^1 + \bar{\rho} dW_t^{\perp}\right), \quad t \geq 0, \tag{6.24}$$

$$dZ_t = \zeta(t,Y_t)dt, \quad t \geq 0. \tag{6.25}$$

Using the Feynman-Kac formula, we deduce that the solution to (6.23) can be represented by

$$f(t,y,z) = E_{\mathbb{Q}_0}^{t,y,z}\left[e^{-\gamma\bar{\rho}^2 g(Y_T,Z_T)}\right]. \tag{6.26}$$

The representation (6.20) thus follows by substituting (6.26) into (6.22). The fact that Q_0 is the minimal martingale measure has been well established [see, for example, 52, 125, 126]. □

Remark 6.4.3. *Under the present stochastic volatility model, the density of an* equivalent local martingale measure *(ELMM)* $\mathbb{Q}$ *is given by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{ -\int_0^T \lambda_s dW_s^1 - \int_0^T \varphi_s dW_s^{\perp} - \frac{1}{2}\int_0^T \left(\lambda_s^2 + \varphi_s^2\right) ds \right\}, \tag{6.27}$$

where $\mathbb{P}$ is the historical measure, λ is the Sharpe ratio process of the stock, and φ is an adapted process satisfying $\int_0^T \varphi_s^2 ds < +\infty$ a.s. [see 12]. We assume that $E[d\mathbb{Q}/d\mathbb{P}] = 1$ so that $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}_T$. A sufficient condition to ensure this is the Novikov condition

$$E\left[\exp\left\{\frac{1}{2}\int_0^T \lambda^2(s, Y_s)ds\right\}\right] < +\infty. \tag{6.28}$$

It is worth noting that the set of all ELMM is one-to-one correspondence to the set Φ of integrands φ. The MMM $\mathbb{Q}_0$ corresponds to $\varphi = 0$ in (6.27). Under $\mathbb{Q}_0$, the discounted stock price is a martingale while the law of the orthogonal martingale measure remains unchanged. For models with continuous price process, it has been shown that $\mathbb{Q}_0$ is the martingale measure minimizing the reverse entropy $H(\mathbb{P}|\mathbb{Q})$ over all ELMM $\mathbb{Q} \in \mathcal{M}$, i.e.

$$\mathbb{Q}_0 = \arg\min_{\mathbb{Q}\in\mathcal{M}} H(\mathbb{P}|\mathbb{Q}) := \arg\min_{\mathbb{Q}\in\mathcal{M}} E_{\mathbb{P}}\left(-\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right). \tag{6.29}$$

We refer the reader to [125, 126] for more discussions.

We are now ready to derive a closed-form formula for the forward indifference price.

Theorem 6.4.4. *The forward indifference price (FIP) for a volatility derivative $G_T = g(Y_T, Z_T)$ is given by $\nu_t(G_T) = v(t, Y_t, Z_t)$, where the pricing function*

$$v(t, y, z) = -\frac{1}{\gamma\bar{\rho}^2}\log E_{\mathbb{Q}_0}^{t,y,z}\left[e^{-\gamma\bar{\rho}^2 g(Y_T, Z_T)}\right], \tag{6.30}$$

for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+$.

Proof. Using the value function (6.16) and (6.18), we rewrite the pricing condition (6.15) as

$$U(t, x) = u(t, x - \nu_t, y, z)\exp\left\{\frac{1}{2}\int_0^t \lambda_s^2 ds\right\}, \tag{6.31}$$

for $(t, x, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. To this end, the forward indifference pricing formula (6.39) follows directly from combining the dynamic performance criterion (6.12) and the value function (6.20). $\square$

From equation (6.23) and the fact that $v(t, y, z) = -\frac{1}{\gamma\bar{\rho}^2} \ln f(t, y, z)$, it follows that the pricing function v solves the quasilinear equation

$$v_t + \frac{1}{2}a^2 v_{yy} + (b - \rho\lambda a)v_y + \zeta v_z = \frac{1}{2}\gamma\bar{\rho}^2 a^2 v_y^2, \tag{6.32}$$

for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+$, with terminal condition $v(T, y, z) = g(y, z)$. Moreover, by Itô calculus, we easily deduce that the indifference price process ν_s satisfies

$$\begin{aligned} d\nu_s =& \rho\lambda a(s, Y_s)v_y(s, Y_s)ds + \frac{1}{2}\gamma\bar{\rho}^2 a^2(s, Y_s)v_y^2(s, Y_s)ds \\ &+ a(s, Y_s)v_y(s, Y_s)(\rho dW_s^1 + \bar{\rho}dW_s^\perp), \end{aligned} \tag{6.33}$$

for $s \geq t$. It follows that the *market price of risk* associated with the non-traded risk component $W^\perp$ is given by

$$\lambda_s^\perp = \frac{1}{2}\gamma\bar{\rho}a(s, Y_s)v_y(s, Y_s), \tag{6.34}$$

for $s \geq t$.

6.4.1 The Optimal Portfolio and Risk Monitoring

In the sequel, we follow the analysis introduced by [111] to construct the risk monitoring strategy and payoff decomposition.

We next recall that the optimal portfolio for the investment problem (6.18) is provided in the feedback form

$$\pi^*(s, x, y) = \frac{1}{\gamma}\frac{\mu(s, y)}{\sigma^2(s, y)} - \rho\frac{a(s, y)}{\sigma(s, y)}v_y(s, y, z),$$

for $s \in [t, T]$.

In absence of the derivative, the optimal portfolio is given by

$$\pi^{0,*}(s, y) = \frac{1}{\gamma}\frac{\mu(s, y)}{\sigma^2(s, y)},$$

for $s \in [t, T]$.

6.5 Backward Indifference Valuation

In the section, we focus on the traditional backward indifference valuation with emphasis on volatility derivatives. The valuation methodology is based on the comparison between two optimal investment problems with and without involving the derivative. In the traditional model, the investor aims to maximize her expected utility of terminal wealth, say, at time T. Consider a *constant absolute risk aversion* (CARA) investor with utility function given by

$$\bar{u}(x) := -e^{-\gamma x}, \quad x \in \mathbb{R}, \tag{6.35}$$

where $\gamma > 0$ is the risk aversion parameter.

The first relevant optimization problem is the classical Merton model. The value function is defined as

$$M(t,x,y) := \sup_{\mathcal{A}} E_t^{x,y}\left[\bar{u}(X_T)\right], \tag{6.36}$$

for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, where the wealth process (X_s) satisfies (6.6) and $\mathcal{A}$ is the set of admissible policies. The second optimal investment problem incorporates the derivative G_T, and the corresponding value function is defined by

$$V(t,x,y,z) := \sup_{\mathcal{A}} E_t^{x,y,z}\left[\bar{u}(X_T + g(Y_T,Z_T))\right], \tag{6.37}$$

for $(t,x,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$.

Using the above value functions, the indifference price for the derivative G_T is then defined as the amount $h(t,x,y,z)$ such that

$$M(t,x,y) = V(t, x - h(t,x,y,z), y, z), \tag{6.38}$$

for $(t,x,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, where the value functions M and V are defined in (6.36) and (6.37), respectively.

Proposition 6.5.1. *The backward indifference price is given by*

$$h(t,y,z) = -\frac{1}{\gamma\bar{\rho}^2} \log E_{\mathbb{Q}}^{t,y,z}\left[e^{-\gamma\bar{\rho}^2 g(Y_T,Z_T)}\right], \tag{6.39}$$

for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+$, *where the pricing measure* $\mathbb{Q}$ *is defined through an Esscher transform*

$$\frac{d\mathbb{Q}}{d\mathbb{Q}_0} = \frac{\exp\left\{-\frac{1}{2}\bar{\rho}^2 \int_t^T \lambda^2(s, Y_s)ds\right\}}{E_{\mathbb{Q}_0}^{t,y}\left[\exp\left\{-\frac{1}{2}\bar{\rho}^2 \int_t^T \lambda^2(s, Y_s)ds\right\}\right]}. \tag{6.40}$$

We remark here that the pricing measure $\mathbb{Q}$ is actually *minimal entropy martingale measure* (MEMM); see [108].

6.6 Applications

In this section, we apply our model and develop numerical procedures to value a variety of volatility derivatives, such as variance and volatility swaps, variance options in European and Asian styles. To make the treatment applicable directly to practice, we focus our attention on the Heston stochastic volatility model, which is popular in the industry [see 58];

6.6.1 Heston Model (Square-root Model)

The Heston stochastic model [see 69] corresponds to choose $\mu(t, Y_t) = \mu(t)$ and $\sigma(t, Y_t) = \sqrt{Y_t}$ in equation (6.3), and $b(t, Y_t) = -\eta(Y_t - \bar{Y})$ and $a(t, Y_t) = \delta\sqrt{Y_t}$ in equation (6.4). These equations then become

$$dS_t = \mu(t)S_tdt + \sqrt{Y_t}S_tdW_t^1, \quad t \geq 0, \tag{6.41}$$

$$dY_t = -\eta(Y_t - \bar{Y})dt + \delta\sqrt{Y_t}dW_t, \quad t \geq 0, \tag{6.42}$$

where η is the speed of reversion of Y_t to its long-term mean $\bar{Y}$, and $W_t = \rho W_t^1 + \bar{\rho}W_t^{\perp}$.

The process followed by the instantaneous variance Y_t may be recognized as a version of the so-called CIR process introduced by [29]. The CIR process and its variants have been widely used to model the interest rate term structure, in particular, the *affine* term structure model.

Unless otherwise specified, we use the well-known BCC parameters [see 7] for the Heston model, which are specified in Table 6.1. We also set the default parameters for risk aversion γ, time to maturity T, and strike price K.

Table 6.1: Heston model parameters.

r	μ	Y_0	$\bar{Y}$	η	δ	ρ	γ	T	K
4.0%	12.7%	0.04	0.04	1.15	39%	-28%	0.1	1.0	0.04

Under the minimal martingale measure $\mathbb{Q}_0$, the dynamics of S, Y and Z are given by

$$dS_t = \sqrt{Y_t}S_t d\tilde{W}_t^1, \quad t \geq 0, \tag{6.43}$$

$$dY_t = \big((\eta\bar{Y} - \rho\mu\delta) - \eta Y_t\big)dt + \delta\sqrt{Y_t}d\tilde{W}_t, \quad t \geq 0, \tag{6.44}$$

where $\tilde{W}^1$ and $\tilde{W}^\perp$ are standard Brownian motions under $\mathbb{Q}_0$, with correlation ρ.

6.6.2 Reciprocal Heston Model (3/2 Model)

The 3/2 model corresponds to choose $\mu(t, Y_t) = \mu(t)$ and $\sigma(t, Y_t) = \sqrt{Y_t}$ in equation (6.3), and $b(t, Y_t) = -(\alpha Y_t - \beta Y_t^2)$ and $a(t, Y_t) = \theta Y_t^{3/2}$ in equation (6.4). That is, these equations become

$$dS_t = \mu(t)S_t dt + \sqrt{Y_t}S_t dW_t^1, \quad t \geq 0,$$

$$dY_t = (\alpha Y_t - \beta Y_t^2)dt + \theta Y_t^{\frac{3}{2}}dW_t, \quad t \geq 0.$$

We observe that the above variance process Y_t is exactly the reciprocal of a CIR process. Thus, it is equivalent to write

$$dS_t = \mu(t)S_t dt + \frac{1}{\sqrt{Y_t}}S_t dW_t^1, \quad t \geq 0, \tag{6.45}$$

$$dY_t = -\eta(Y_t - \bar{Y})dt + \delta\sqrt{Y_t}dW_t, \quad t \geq 0, \tag{6.46}$$

where the process Y_t becomes the reciprocal variance process.

We will see that this latter specification gives advantage for a unified numerical treatment. This also explained why the 3/2 model is sometimes called the reciprocal Heston model. As argued by [65], the reciprocal CIR process is reasonable model for stochastic volatility (variance) because of its positivity and mean reverting. The reciprocal CIR process has used by [1] in modeling term structure, which yields a closed-form solution for bond prices.

Unless otherwise specified, the parameters for the reciprocal Heston model (6.45) and (6.46) are set in Table 6.2.

Table 6.2: Reciprocal Heston model parameters.

r	μ	Y_0	$\bar{Y}$	η	δ	ρ	γ	T	K
4.0%	12.7%	32.26	28.57	1.15	39%	-28%	0.1	1.0	0.04

Under the minimal martingale measure $\mathbb{Q}_0$, the dynamics of S, Y and Z are given by

$$dS_t = \frac{1}{\sqrt{Y_t}} S_t d\tilde{W}_t^1, \quad t \geq 0, \tag{6.47}$$

$$dY_t = \big(\eta\bar{Y} - (\eta + \rho\mu\delta)Y_t\big)dt + \delta\sqrt{Y_t}d\tilde{W}_t, \quad t \geq 0, \tag{6.48}$$

where $\tilde{W}^1$ and $\tilde{W}^\perp$ are standard Brownian motions under $\mathbb{Q}_0$, with correlation ρ.

6.6.3 Volatility Derivatives

We focus on the applications to variance and volatility swaps. The discounted payoff of a variance swap is specified by

$$G_T = e^{-r(T-t)} \int_t^T \sigma^2(s, Y_s) ds, \tag{6.49}$$

and that of a volatility swap is given by

$$G_T = e^{-r(T-t)} \left(\int_t^T \sigma^2(s, Y_s) ds \right)^{\frac{1}{2}}. \tag{6.50}$$

We also consider European options and Asian options on realized variance. For example, the discounted payoff of a European variance call is given by

$$G_T = e^{-r(T-t)} \left(\int_t^T \sigma^2(s, Y_s) ds - K \right)^+, \tag{6.51}$$

which corresponds to choose $g(y,z) = (z-K)^+$ and $\zeta(t,y) = \sigma^2(t,y)$ in (6.1). The payoff of an Asian style variance option is given by

$$G_T = e^{-r(T-t)} \left(\sigma^2(T, Y_T) - \frac{1}{T-t} \int_t^T \sigma^2(s, Y_s) ds \right)^+, \tag{6.52}$$

which corresponds to choose $g(y,z) = \big(\sigma^2(T,y) - z\big)^+$ and $\eta(t,y) = \sigma^2(t,y)$ in (6.1).

6.7 Numerical Treatments

In this section, we develop a numerical procedure based on Monte Carlo simulation to value a generic volatility derivative considering its path-dependent feature. The essential step in the simulation is to generate the paths of the CIR process given by

$$dY_t = (A(t) - B(t)Y_t)dt + C\sqrt{Y_t}d\tilde{W}_t, \quad t \geq 0, \tag{6.53}$$

with $Y_0 \geq 0$. It is well known that the above SDE (6.53) has a unique nonnegative solution provided that $A \geq 0$ and $C \geq 0$ [see 117]. If, in addition, we impose condition $A > C^2$, then the process is always positive [see 87]. Note also that the process is mean-reverting when $B > 0$, which is a desirable property for modeling stochastic volatility. The challenge is to produce positive paths.

6.7.1 Monte Carlo Simulations

Milstein Discretization. It is straightforward to discretize the paths using a Milstein scheme. Specifically, by Ito-Taylor expansion, we have

$$Y_{t+\Delta t} - Y_t = (A - BY_t)\Delta t + C\sqrt{Y_t}\Delta W + \frac{C^2}{4}(\Delta W^2 - \Delta t).$$

It follows that

$$Y_{t+\Delta t} = \left(\sqrt{Y_t} + \frac{C}{2} w \sqrt{\Delta t} \right)^2 + \left(A - \frac{C^2}{4} - BY_t \right) \Delta t, \tag{6.54}$$

where $w \sim N(0,1)$. The problem of Milstein scheme is that it does not guarantee a positive path, though it reduces significantly the negativity problem if comparing to the Euler scheme. In the sequel, we follow [2] to develop schemes the guarantee the positivity of the CIR process.

The First Implicit Schemes. We first consider the following discretization

$$Y_{t+\Delta t} - Y_t = (A - BY_{t+\Delta t})\Delta t + C\sqrt{Y_{t+\Delta t}}\Delta W - \frac{C}{2}\Delta t,$$

which follows the quadratic equation

$$(1 + B\Delta t)Y_{t+\Delta t} - Cw\sqrt{\Delta t} \cdot \sqrt{Y_{t+\Delta t}} - \left[Y_t + \left(A - \frac{C^2}{2}\right)\Delta t\right] = 0.$$

The only positive root yields the first implicit scheme

$$\sqrt{Y_{t+\Delta t}} = \frac{Cw\sqrt{\Delta t} + \sqrt{C^2 w^2 \Delta t + 4(1 + B\Delta t)\left[Y_t + \left(A - \frac{C^2}{2}\right)\Delta t\right]}}{2(1 + B\Delta t)} \tag{6.55}$$

The Second Implicit Schemes. By the Ito formula, we calculate

$$d\sqrt{Y_t} = \left(\frac{4A - C^2}{8\sqrt{Y_t}} - \frac{B}{2}\sqrt{Y_t}\right)dt + \frac{C}{2}dW_t, \quad t \geq 0.$$

Implicit in the drift gives

$$\sqrt{Y_{t+\Delta t}} - \sqrt{Y_t} = \left(\frac{4A - C^2}{8\sqrt{Y_{\Delta t}}} - \frac{B}{2}\sqrt{Y_{\Delta t}}\right)\Delta t + \frac{C}{2}\Delta W,$$

which implies the following quadratic equation in $\sqrt{Y_{t+\Delta t}}$

$$\left(1 + \frac{B}{2}\Delta t\right)Y_{t+\Delta t} - \sqrt{Y_{t+\Delta t}}\left(\sqrt{Y_t} + \frac{C}{2}w\sqrt{\Delta t}\right) - \frac{4A - C^2}{8\sqrt{Y_{t+\Delta t}}}\sqrt{\Delta t} = 0.$$

The only positive root is then given by

$$\sqrt{Y_{t+\Delta t}} = \frac{\sqrt{Y_t} + \frac{C}{2}w\sqrt{\Delta t} + \sqrt{\left(\sqrt{Y_t} + \frac{C}{2}w\sqrt{\Delta t}\right)^2 + (2 + B\Delta t)\left(A - \frac{C^2}{4}\right)\Delta t}}{2 + B\Delta t}, \tag{6.56}$$

which provides the second implicit scheme.

Explicit Schemes. Applying Taylor expand to the above two implicit schemes, we obtain the following explicit schemes

$$Y_{t+\Delta t} = \left(\left(1 - \frac{B}{2}\Delta t\right)\sqrt{Y_t} + \frac{Cw\sqrt{\Delta t}}{2 - B\Delta t}\right)^2 + \left(A - \frac{C^2}{4}\right)^2 \Delta t + k\left(\Delta W_t^2 - \Delta t\right),$$

for $0 \le k \le A - \frac{C^2}{4}$, where the factor $1 - B\Delta t/2$ can be replaced by $\sqrt{1 - B\Delta t}$. It follows that

$$Y_{t+\Delta t} = \left(\left(1 - \frac{B}{2}\Delta t\right)\sqrt{Y_t} + \frac{Cw\sqrt{\Delta t}}{2 - B\Delta t}\right)^2 + \left(A - \frac{C^2}{4} - k + kw^2\right)^2 \Delta t, \quad (6.57)$$

or, by using the square-root factor instead, that

$$Y_{t+\Delta t} = \frac{\left(2(1 - B\Delta t)\sqrt{Y_t} + Cw\sqrt{\Delta t}\right)^2}{4(1 - B\Delta t)} + \left(A - \frac{C^2}{4} - k + kw^2\right)^2 \Delta t, \quad (6.58)$$

for $0 \le k \le A - \frac{C^2}{4}$.

6.7.2 Laplace Transform

In some special cases, we may apply the formula by [65] to compute the indifference prices. Let $\alpha = \frac{2A^2}{C} - 1$ and $\beta = \frac{2B}{C^2}$, and assume that $d_1 \ge -\frac{B^2}{2C^2}$ and $d_2 \ge -\frac{\alpha^2 C^2}{8}$. Then the Laplace transform

$$\begin{aligned} E^y\left[e^{-d_1\int_0^T Y_s ds - d_2\int_0^T Y_s^{-1} ds}\right] &= \exp\left\{-y\left(\phi_1 - \frac{\theta\phi_1}{\theta - \phi_1}e^{-(B+\phi_1 C^2)T}\right)\right\} \\ &\quad + e^{(-A\phi_1 + B\phi_2 + C^2\phi_1\phi_2)T} y^{\phi_2} \frac{\theta^{\alpha+2\phi_2+1}}{(\theta - \phi_1)^{\alpha+\phi_2+1}} \\ &\quad + \frac{\alpha + \phi_2 + 1}{\alpha + 2\phi_2 + 1} {}_1\mathrm{F}_1\left(\phi_2, \alpha + 2\phi_2 + 1; -\frac{\theta^2 y e^{-(B+\phi_1 C^2)T}}{\theta - \phi_1}\right), \end{aligned}$$

where the ${}_1\mathrm{F}_1$ is the confluent hypergeometric function, and

$$\phi_1 = \frac{1}{2}\left(\sqrt{\beta^2 + \frac{8d_1}{C^2}} - \beta\right), \quad \phi_2 = \frac{1}{2}\left(\sqrt{\alpha^2 + \frac{8d_2}{C^2}} - \alpha\right), \quad (6.59)$$

and

$$\theta = (\beta + 2\phi_1) \Big/ \left(1 - e^{-(B+\phi_1 C^2)T}\right). \tag{6.60}$$

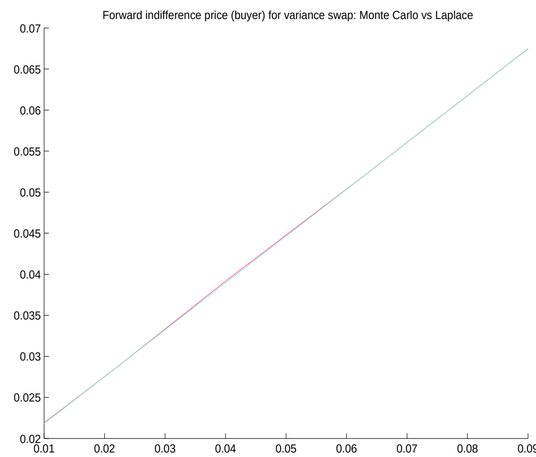
6.7.3 Numerical Examples

We apply the above numerical schemes to compute the indifference prices (BFIP, SFIP, BBIP, and SBIP) of variance swaps. We use the Heston model (6.41) and (6.42) with the BCC parameters given in Table 6.1. The initial level Y_0 of the variance process ranges from 0.01 to 0.09. The numerical results for the second implicit scheme are shown in Table 6.3. The simulation is based on 10^4 (5000 plus 5000 antithetic) sample paths with a time step equal to 10^{-3}. In addition, Figure 6.1 plots the buyer's forward and backward indifference prices as functions of Y_0.

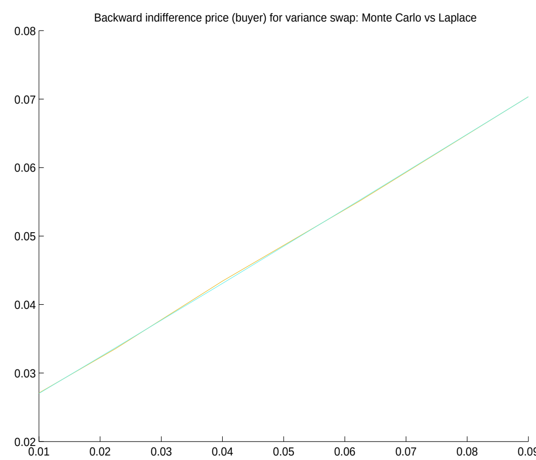
Table 6.3: Forward and backward indifference prices of variance swaps.

	Y_0	Value	(s.e.)	95% C.I.	True Value	Rel.Error
BFIP	0.0100	0.0219	(0.0001)	[0.0217, 0.0221]	0.0218	0.3043%
SFIP	0.0100	0.0219	(0.0001)	[0.0217, 0.0221]	0.0218	0.3050%
BBIP	0.0100	0.0271	(0.0002)	[0.0267, 0.0275]	0.0270	0.3001%
SBIP	0.0100	0.0271	(0.0002)	[0.0267, 0.0275]	0.0270	0.2999%
BFIP	0.0225	0.0289	(0.0001)	[0.0287, 0.0292]	0.0290	-0.0348%
SFIP	0.0225	0.0290	(0.0001)	[0.0287, 0.0293]	0.0290	-0.0359%
BBIP	0.0225	0.0335	(0.0002)	[0.0330, 0.0341]	0.0337	-0.4019%
SBIP	0.0225	0.0336	(0.0002)	[0.0331, 0.0341]	0.0337	-0.4025%
BFIP	0.0400	0.0392	(0.0001)	[0.0389, 0.0394]	0.0389	0.5496%
SFIP	0.0400	0.0392	(0.0001)	[0.0390, 0.0394]	0.0390	0.5547%
BBIP	0.0400	0.0434	(0.0002)	[0.0429, 0.0439]	0.0431	0.7904%
SBIP	0.0400	0.0435	(0.0002)	[0.0430, 0.0440]	0.0431	0.7952%
BFIP	0.0625	0.0518	(0.0001)	[0.0515, 0.0521]	0.0518	0.0004%
SFIP	0.0625	0.0518	(0.0001)	[0.0516, 0.0521]	0.0518	-0.0002%
BBIP	0.0625	0.0552	(0.0002)	[0.0546, 0.0557]	0.0553	-0.2298%
SBIP	0.0625	0.0552	(0.0002)	[0.0547, 0.0558]	0.0553	-0.2297%
BFIP	0.0900	0.0675	(0.0001)	[0.0673, 0.0677]	0.0675	0.0182%
SFIP	0.0900	0.0676	(0.0001)	[0.0674, 0.0678]	0.0676	0.0176%
BBIP	0.0900	0.0704	(0.0002)	[0.0700, 0.0707]	0.0703	0.0326%
SBIP	0.0900	0.0704	(0.0002)	[0.0701, 0.0708]	0.0704	0.0319%

This table shows the forward and backward indifference prices, estimated by Monte Carlo simulation. The simulation is based on 10^4 (5000 plus 5000 antithetic) sample paths from the underlying CIR process. The standard errors (s.e.) of the simulation estimates are given in parentheses and the 95% confidence intervals (C.I.) in brackets. The true values are computed using formula (6.7.2).



(a) Buyer's Forward Indifference Price



(b) Buyer's Backward Indifference Price

Figure 6.1: Forward and backward indifference prices as functions of spot price Y_0

Chapter 7

Conclusion and Future Research

This dissertation contributes to the derivatives pricing in incomplete financial markets. We first developed a valuation approach based on equilibrium arguments from the perspective of option market makers and financial intermediaries. This approach produces a pricing notion of *competition-based price*. We analyzed such price in both a Markovian diffusion and a semimartingale setting.

In both settings, we consider a financial market with an arbitrary number of market makers. We model their competition and apply tools from stochastic control and convex duality to analyze the competitive equilibrium. The resulting pricing measure is characterized as the *minimal entropy martingale measure* (MEMM) with respect to a *new prior*. This new prior depends on the aggregate demand and inventory of the derivatives and is characterized as an *Esscher transform* of the historical measure. In the diffusion setting, the pricing measure is explicitly constructed.

We further investigated the price effects of demand and inventory. Our model shows that the price of a derivative is increasing with the demand of any derivative in the derivative market. The increasing rate is proportional to the covariance between the unhedgeable parts of the payoffs of the associated derivatives, calculated under the competitive pricing measure. This result provides a possible resolution for the *options-pricing puzzles*, namely, index options appear to be expensive and low-moneyness options seem to be especially expensive comparing to other individual equity options.

The notion of competition-based price turns out to be the natural generalization of the *marginal utility price*. We also investigated the various properties of the competition-based price. In addition, we applied our model to price a variety of volatility derivatives and developed the associated numerical schemes to compute the competitive prices under Heston stochastic volatility model.

In the second part, we applied the utility-based approach to value a variety of options with staging structure and sequential decisions in incomplete markets. Examples include multiple claims with distinct maturities, installment call and put options, compound options, ASX installment warrants, and staged financing of venture capital investments.

For a generic installment option written on both tradable and non-traded risk factors, we have constructed the indifference price in terms of a quasilinear PDE and analyzed the associated risk monitoring strategies. In particular, we obtained an explicit nonlinear pricing formula for options contingent exclusively on the non-traded risk factor. We further developed a Monte Carlo procedure, based on the least-squares approximation, to compute the indifference prices. Through various numerical experiments, we investigated the dependence of the indifference price and the probability of early termination on the installment structure (such as the level and the number of installments) and other market parameters (such as initial asset price, time to maturity, risk aversion parameter, and correlation coefficient). The model was further applied to analyze the dilution effect of ASX installment warrants and the staged financing of venture capital investments.

Moreover, we introduced a *foresighted valuation* framework to incorporate the investors' private information into their valuation and hedging strategies. Such information may include both their *ex-ante* risk exposure and *ex-post* investment opportunities. Finally, we adopt the recently developed *dynamic performance criteria* to price volatility derivatives. We develop numerical schemes for the computation of the *forward and backward indifference prices* in models of Heston and reciprocal-Heston type.

For future research in the equilibrium approach, we plan to investigate more applications such as credit and insurance markets, which, in general, requires to analyze the jump-diffusion model. The second possible direction is to calibrate the valuation model to the real market data, and analyze the *options-pricing puzzles* quantitatively. Another interesting problem is to use more general utility, for example, the dynamic utilities (dynamic performance criteria) recently developed by Musiela and Zariphopoulou. It is also possible to introduce demand elasticity, bidask spread, or constraints (such as borrowing, endowment, wealth, etc.) into the model.

For the indifference approach, the results developed herein can be applied or extended to analyze other sequential investments, installment loans, and many corporate liabilities with sequential opportunities (such as equity with coupon bearing bonds, bonds with extended maturities, and callable bonds with coupon payments). Another possible direction would be the combination of static and dynamic hedging strategies for installment options. A question of independent interest is to investigate how to apply the present approach to analyze the optimal design of installment warrants and agency problems arising in venture capital.

Appendix A

Proofs and Derivations

For convenience of reference, we provide in this appendix detail proofs and derivations for certain results, whose proofs are skipped in the main text.

A.1 Proofs of Chapter 3

Proof of Theorem 3.3.4.

(i) By the definition of v, we clearly have for any $\alpha, \beta \in \mathbb{R}^n$,

$$f(Q(\beta);\beta) - f(Q(\beta);\alpha) \leq v(\beta;G) - v(\alpha;G) \leq f(Q(\alpha);\beta) - f(Q(\alpha);\alpha), \quad \text{(A.1)}$$

which yields that

$$\begin{aligned}
|v(\beta;G) - v(\alpha;G)| &\leq \max\left\{\left|f(Q(\alpha);\beta) - f(Q(\alpha);\alpha)\right|, \left|f(Q(\beta);\beta) - f(Q(\beta);\alpha)\right|\right\} \\
&\leq \sup_{Q\in\mathbb{P}_f} |f(Q;\beta) - f(Q;\alpha)| \\
&= \sup_{Q\in\mathbb{P}_f} \left|(\beta-\alpha)\cdot E^Q[G]\right| \\
&\leq |(\beta-\alpha)| \sup_{Q\in\mathbb{P}_f} E^Q[|G|].
\end{aligned}$$

Since G is bounded, $\sup_{Q\in\mathbb{P}_f} E^Q[|G|] \leq K$ for some constant K. It thus follows that

$$\lim_{\beta\to\alpha} |v(\beta;G) - v(\alpha;G)| \leq K \lim_{\beta\to\alpha} |(\beta-\alpha)| = 0,$$

which proves the (absolute) continuity of v in α.

(ii) We next recall that

$$v(\alpha; G) = f(Q(\alpha); \alpha) = H(Q(\alpha)|P) + E^{Q(\alpha)}[\alpha \cdot G],$$

which yields that

$$H(Q(\beta)|P) - H(Q(\alpha)|P) = v(\beta; G) - v(\alpha; G) - \left(E^{Q(\beta)}[\beta \cdot G] - E^{Q(\alpha)}[\alpha \cdot G]\right).$$

It further follows from the identity (3.11) that

$$\begin{aligned} H(Q(\beta)|P_\alpha) - H(Q(\alpha)|P_\alpha) &= H(Q(\beta)|P) - H(Q(\alpha)|P) + E^{Q(\beta)}[\alpha \cdot G] - E^{Q(\alpha)}[\alpha \cdot G] \\ &= v(\beta; G) - v(\alpha; G) - (\beta - \alpha) \cdot E^{Q(\beta)}[G]. \end{aligned} \tag{A.2}$$

Thus the continuity of $H(Q(\beta)|P_\alpha)$ follows from the continuity of v and boundness of $E^{Q(\beta)}[G]$ since

$$\lim_{\beta \to \alpha} |H(Q(\beta)|P_\alpha) - H(Q(\alpha)|P_\alpha)| \leq \lim_{\beta \to \alpha} |v(\beta; G) - v(\alpha; G)| + K \lim_{\beta \to \alpha} |\beta - \alpha| = 0.$$

Now by Lemma 3.3.3 [31, Theorem 2.2], we have

$$H(Q(\beta)|P_\alpha) \geq H(Q(\beta)|Q(\alpha)) + H(Q(\alpha)|P_\alpha),$$

since $Q(\alpha)$ minimize entropy $H(Q|P_\alpha)$. This and the continuity of $H(Q(\beta)|P_\alpha)$ further imply that

$$\lim_{\beta \to \alpha} H(Q(\beta)|Q(\alpha)) \leq \lim_{\beta \to \alpha} |H(Q(\beta)|P_\alpha) - H(Q(\alpha)|P_\alpha)| = 0.$$

We thus conclude that $Q(\beta)$ converges to $Q(\alpha)$ in entropy and thus in total variation, i.e.

$$\lim_{\beta \to \alpha} E^{Q(\beta)}[G] = E^{Q(\alpha)}[G],$$

which proves the continuity of $E^{Q(\alpha)}[G]$. The continuity of $H(Q(\alpha)|P)$ further follows from (3.15) and the continuity of $v(\alpha; G)$ and $E^{Q(\alpha)}[G]$.
(iii) We next establish the differentiability of v. From (A.1) and the definition of f,

we obtain

$$(\beta-\alpha)\cdot E^{Q(\beta)}[G] \leq v(\beta;G)-v(\alpha;G) \leq (\beta-\alpha)\cdot E^{Q(\alpha)}[G],$$

and thus

$$(\beta-\alpha)\left(E^{Q(\beta)}[G]-E^{Q(\alpha)}[G]\right) \leq v(\beta;G)-v(\alpha;G)-(\beta-\alpha)\cdot E^{Q(\alpha)}[G] \leq 0.$$

Therefore, by the continuity of $E^{Q(\alpha)}[G]$ we conclude that

$$\lim_{\beta\to\alpha} \frac{\left|v(\beta;G)-v(\alpha;G)-(\beta-\alpha)\cdot E^{Q(\alpha)}[G]\right|}{|\beta-\alpha|} \leq \lim_{\beta\to\alpha}\left|E^{Q(\beta)}[G]-E^{Q(\alpha)}[G]\right| = 0,$$

which shows that v is continuously differentiable and its gradient is given by (3.21). □

Proof of Theorem 3.3.4: continuity of $H(Q(\beta)|P_\alpha)$.

In what follows, we provide another proof for the continuity of $H(Q(\beta)|P_\alpha)$. From the identity (3.11), we find that

$$\begin{aligned}&\big(H(Q(\beta)|P_\alpha)-H(Q(\alpha)|P_\alpha)\big)+\big(H(Q(\alpha)|P_\beta)-H(Q(\beta)|P_\beta)\big)\\&=(\alpha-\beta)\cdot\left(E^{Q(\beta)}[G]-E^{Q(\alpha)}[G]\right).\end{aligned}$$

Observing that the right hand side is a sum of two positive terms, we conclude that

$$0 \leq H(Q(\beta)|P_\alpha)-H(Q(\alpha)|P_\alpha) \leq (\alpha-\beta)\cdot\left(E^{Q(\beta)}[G]-E^{Q(\alpha)}[G]\right).$$

Since $\sup_{Q\in\mathbb{P}_f} E^Q[|G|] \leq K$, it follows that

$$\lim_{\beta\to\alpha}|H(Q(\beta)|P_\alpha)-H(Q(\alpha)|P_\alpha)| \leq 2K\lim_{\beta\to\alpha}|\beta-\alpha| = 0,$$

which proves the continuity of $H(Q(\beta)|P_\alpha)$. □

Proof of Theorem 3.4.1: existence and uniqueness

We provide another proof of the existence and uniqueness by using Theorem 3.3.5. The existence of ϑ_i have been well established in the duality arguments. For the existence and uniqueness of q_i, we combine (3.8), (3.9), and (3.16) to write

$$u(x, \alpha; G) = -\exp\left\{-(x + v(\alpha; G))\right\}, \tag{A.3}$$

where v is the value function of the dual problem given by (3.15). Therefore, we see that

$$\max_{q_i \in \mathbb{R}^n} u\left(\gamma_i(x_i - q_i \cdot p), \gamma_i q_i; G\right) = -e^{-\gamma_i x_i} \exp\left\{-\max_{q_i \in \mathbb{R}^n}\Big(v(\gamma_i q_i; G) - \gamma_i q_i \cdot p\Big)\right\}.$$

and thus the existence and uniqueness of q_i follow from the strict concavity of $v(\alpha; G)$ as shown in Theorem 3.3.5. $\square$

A.2 Proofs of Chapter 4

Proof of Proposition 4.2.1

By the Dynamic Programming Principle (DPP), we see that

$$\begin{aligned} M(t, S, x) &= \sup_{\pi \in \mathcal{A}[t,s]} E\left[M(s, S_s, X_s^{0,\pi}) \middle| S_t = S, X_t^0 = x\right] \\ &= E\left[M(s, S_s, X_s^{0,*}) \middle| S_t = S, X_t^{0,*} = x\right], \end{aligned}$$

for any $s \in [t, T]$, which means that the value process $M(s, S_s, X_s^{0,*})$ is a local martingale provided the existence of optimal policy $\pi^{0,*}$. Assuming the smoothness of M, we then apply the Itô's formula to calculate

$$\begin{aligned} dM(s, S_s, X_s^{0,*}) &= \left(M_t + \frac{1}{2}\sigma^2(\pi_s^{0,*})^2 M_{xx} + \pi_s^{0,*}\left(\sigma^2 S_s M_{Sx} + \mu M_x\right) + \mathcal{L}^{(S)} M\right) ds \\ &\quad + \left(\sigma \pi_s^{0,*} M_x + \sigma S_s M_S\right) dW_s^1, \end{aligned}$$

where we've compressed the arguments (s, S_s) of μ, σ, and $(s, S_s, X_s^{0,*})$ of M for simplicity. The above DPP implies that the conditional expectation

$$E_t^{S,x} \int_t^s \left(M_t + \frac{1}{2}\sigma^2 (\pi_u^{0,*})^2 M_{xx} + \pi_u^{0,*} \left(\sigma^2 S_u M_{Sx} + \mu M_x \right) + \mathcal{L}^{(S)} M \right) du = 0,$$

for any $s \in [t, T]$. Therefore, by multiplying $\frac{1}{s-t}$ on both sides of the previous equation and further sending $s \to t$, we conclude that the smooth value function if exists must satisfy the HJB equation (4.12).

We next argue the probabilistic representation. By plugging into equation (4.12) the optimal control policy

$$\pi^{0,*} = -\frac{\mu M_x + \sigma^2 S M_{Sx}}{\sigma^2 M_{xx}}, \tag{A.4}$$

the HJB equation can be rewritten as

$$M_t - \frac{\left(\mu M_x + \sigma^2 S M_{Sx}\right)^2}{2\sigma^2 M_{xx}} + \mathcal{L}^{(S)} M = 0,$$

for $(t, S, x) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}$, with terminal condition $M(T, S, x) = -e^{-\gamma x}$. Substituting the ansatz transformation $M(t, S, x) = -\exp\{-\gamma x - f(t, S)\}$ into the previous equation, we find that $f(t, S)$ solves

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + \frac{1}{2}\lambda^2 = 0, \tag{A.5}$$

for $(t, S) \in [0, T] \times \mathbb{R}^+$, with terminal condition $f(T, S) = 0$. Using the Feynman-Kac connection, we conclude that f can be represented as expectation (4.14) under martingale measure $\mathbb{Q}$ specified by (4.15). Indeed, the Girsanov's theorem implies that the process $\tilde{W}_t := W_t^1 + \int_0^t \lambda_s ds, t \in [0, T]$, is a standard Brownian motion under measure $\mathbb{Q}$. Applying Itô's formula to $f(s, S_s)$ and combining equation (A.5) yields

$$0 = f(T, S_T) = f(t, S_t) - \frac{1}{2}\int_t^T \lambda_s^2 ds + \int_t^T \sigma(s, S_s) S_s f_S(s, S_s) d\tilde{W}_s.$$

The representation (4.14) thus follows by taking conditional expectation under measure $\mathbb{Q}$ on both sides of the above equality. Note that $f(t, S) = \gamma m(t, S)$.

Furthermore, by substituting (4.13) into the (A.4), we obtain the optimal policy $\Pi_s^{0,*}$ in feedback form $\pi_s^{0,*}(s, S_s, X_s^{0,*})$ given by (4.16), and the optimal wealth process (4.17) is then follows hereby. Finally, the classical verification results (see for example Theorem IV.3.1 in Fleming and Soner, 1993) ensure the optimality of our solution. $\Box$

Derivation of HJB equation (4.20)

Starting from (4.19), we've seen that the value function is given by

$$V^{\mathcal{C}_i}(t, S, x, y) = \sup_{\pi \in \mathcal{A}[t, T_i]} E\left[V^{\mathcal{C}_{i-1}}(T_i, S_{T_i}, X_{T_i} + g(S_{T_i}, Y_{T_i}))\middle| S_t = S, X_t = x, Y_t = y\right],$$

for $t \in [T_{i+1}, T_i]$, which clearly verifies the terminal condition (4.21). Assuming the smoothness of V, we then apply the Itô's formula to calculate

$$\begin{aligned} dV(s, S_s, X_s^*, Y_s) &= \left(V_t + \frac{1}{2}\sigma^2(\pi_s^*)^2 V_{xx} + \pi_s^*\left(\sigma^2 S_s V_{Sx} + \rho\sigma a V_{xy} + \mu V_x\right) + \mathcal{L}^{(S,y)} V\right) ds \\ &\quad + \left(\sigma \pi_s^* V_x + \sigma S_s V_S\right) dW_s^1 + a V_y dW_s, \end{aligned}$$

where we've suppressed the arguments (s, S_s) of μ, σ, b, a, and (s, S_s, X_s^*, Y_s) of V for simplicity. The Dynamic Programming Principal further implies that the value process $V(s, S_s, X_s^*, Y_s)$ is a local martingale provided the existence of optimal policy π^*. Thus the conditional expectation

$$E_t^{S,x,y} \int_t^s \left(V_t + \frac{1}{2}\sigma^2(\pi_u^*)^2 V_{xx} + \pi_u^*\left(\sigma^2 S_u V_{Sx} + \rho\sigma a V_{xy} + \mu V_x\right) + \mathcal{L}^{(S,y)} V\right) du = 0,$$

for any $s \in [t, T_i]$. Therefore, by multiplying on both sides $\frac{1}{s-t}$, and then sending $s \to t$, we conclude that the smooth value function if exists must satisfy the HJB equation (4.20). $\Box$

Proof of Theorem 4.2.2

By plugging into equation (4.20) the optimal control policy

$$\pi^* = -\frac{\mu V_x + \rho\sigma a V_{xy} + \sigma^2 S V_{Sx}}{\sigma^2 V_{xx}}, \tag{A.6}$$

we rewrite the HJB equation as

$$V_t - \frac{\left(\mu V_x + \rho\sigma a V_{xy} + \sigma^2 S V_{Sx}\right)^2}{2\sigma^2 V_{xx}} + \mathcal{L}^{(S,y)} V = 0,$$

for $t \in (T_{i+1}, T_i]$, with terminal condition (4.21). Thus, by substituting the ansatz form $V(t, S, x, y) = -\exp\{-\gamma(x + \phi(t, S, y))\}$ into the previous equation, we derive the quasilinear equation for ϕ.

$$\phi_t + \tilde{\mathcal{L}}^{(S,y)}\phi - \frac{1}{2}\gamma(1-\rho^2)a^2\phi_y^2 + \frac{1}{2\gamma}\lambda^2 = 0, \tag{A.7}$$

for $t \in (T_{i+1}, T_i]$, with terminal condition $\phi(T_i, S, y) = \phi(T_i^+, S, y) + c_i(S, y)$. To this end, the representation (4.22) of value function $V^{\mathcal{C}_n}$ can be verified by using the δ-function for the pasting condition. We next substitute (4.22) into (A.6) to construct the optimal feedback policy (4.24). Therefore, the optimality of control policy $\Pi_s^* = \pi_s^*(s, S_s, X_s^*, Y_s)$ follows from the standard verification results (see for example Theorem IV.3.1 in Fleming and Soner, 1993). $\square$

Proof of Proposition 4.2.3

In Proposition 4.2.3, we consider the buyer's valuation problem for a claim $c(Y_\tau)$ written only on the nontraded asset Y, where the stock dynamics is assumed to have deterministic coefficients: $\mu(t, S) = \mu(t)$ and $\sigma(t, S) = \sigma(t)$. The value function V^c is then defined by

$$V^c(t, x, y) := \sup_{\pi \in \mathcal{A}_{[t,\tau]}} E\left[M(\tau, X_\tau + c(Y_\tau)) \middle| X_t = x, Y_t = y\right]$$

with terminal condition $V^c(\tau, x, y) = M(\tau, x + c(y))$, where $\mathcal{A}_{[t,\tau]}$ stands for the admissible policies, M is the Merton's value function given by (4.13), (Y_s) satisfies (4.2), and the wealth process (X_s) is given by

$$dX_s = \mu(s)\pi_s ds + \sigma(s)\pi_s dW_s^1, \tag{A.8}$$

for $s \in [t, \tau]$. To simplify presentation, we suppress the arguments of functions μ, σ, b, a, and denote $b_s = b(s, Y_s)$ and $a_s = a(s, Y_s)$ in the following derivation.

Derive HJB Equation for V^c. By the Dynamic Programming Principle

(DPP), we have

$$
\begin{aligned}
V^c(t,x,y) &= \sup_{\pi\in\mathcal{A}_{[t,s]}} E\left[V^c(s,X_s^\pi,Y_s)\middle|X_t=x,Y_t=y\right] \\
&= E\left[V^c(s,X_s^*,Y_s)\middle|X_t^*=x,Y_t=y\right],
\end{aligned}
$$

for any $s\in[t,\tau]$, which shows that the value process $V^c(s,X_s^*,Y_s)$, $s\in[t,\tau]$, is a local martingale provided the existence of optimal policy π^*. Assuming the smoothness of V^c, we apply Itô's formula to calculate

$$
\begin{aligned}
dV^c(s,X_s^*,Y_s) &= \left(V_t^c+\mu\pi_s^*V_x^c+\frac{1}{2}\sigma^2(\pi_s^*)^2V_{xx}^c+\rho\sigma a_s\pi_s^*V_{xy}^c+b_sV_y^c+\frac{1}{2}a_s^2V_{yy}^c\right)ds \\
&\quad +\sigma\pi_s^*V_x^c dW_s^1+a_sV_y^c dW_s,
\end{aligned}
$$

for any $s\in[t,\tau]$, where we drop the arguments (s,X_s^*,Y_s) of V^c for simplicity. That is,

$$
\begin{aligned}
V^c(s,X_s^*,Y_s) &= V^c(t,X_t^*,Y_t)+\int_t^s\sigma\pi_u^*V_x^c dW_u^1+\int_t^s a_uV_y^c dW_u \\
&\quad +\int_t^s\left(V_t^c+\mu\pi_u^*V_x^c+\frac{1}{2}\sigma^2(\pi_u^*)^2V_{xx}^c+\rho\sigma a_u\pi_u^*V_{xy}^c+b_uV_y^c+\frac{1}{2}a_u^2V_{yy}^c\right)du.
\end{aligned}
$$

The above DPP implies that the conditional expectation

$$
E_t^{x,y}\int_t^s\left(V_t^c+\mu\pi_u^*V_x^c+\frac{1}{2}\sigma^2(\pi_u^*)^2V_{xx}^c+\rho\sigma a_u\pi_u^*V_{xy}^c+b_uV_y^c+\frac{1}{2}a_u^2V_{yy}^c\right)du=0,
$$

for any $s\in[t,\tau]$. Therefore, we multiply both sides by $\frac{1}{s-t}$ and further send $s\to t$ to derive the HJB equation for value function V^c:

$$
V_t+\sup_{\pi\in\mathcal{A}_{[t,\tau]}}\left(\frac{1}{2}\sigma^2\pi^2V_{xx}+\pi\left(\mu V_x+\rho\sigma aV_{xy}\right)\right)+\frac{1}{2}a^2V_{yy}+bV_y=0, \tag{A.9}
$$

for $(t,y)\in[0,\tau]\times\mathbb{R}$, with terminal condition

$$
V(\tau,x,y)=-\exp\left\{-\gamma x-\gamma c(y)-\frac{1}{2}\int_\tau^T\lambda^2(s)ds\right\},
$$

where $\lambda(s)=\mu(s)/\sigma(s)$ denotes the *Sharpe ratio* of the stock. Note that we've skipped the superscript c for simplifying the notation.

Linearize the HJB Equation. We observe that the second term of the HJB equation (A.9) is a quadratic form of π. It achieves its maximum value at

$$\pi^* = -\frac{\mu V_x + \rho\sigma a V_{xy}}{\sigma^2 V_{xx}}. \tag{A.10}$$

By substituting π^* into (A.9), the HJB equation becomes

$$V_t - \frac{(\mu V_x + \rho\sigma a V_{xy})^2}{2\sigma^2 V_{xx}} + \frac{1}{2}a^2 V_{yy} + bV_y = 0. \tag{A.11}$$

Using the scaling property of the utility function and the structure of the wealth dynamics equation (4.5), we postulate a candidate solution $V(t,x,y) = -e^{-\gamma x}F(t,y)$. Plugging it into (A.11) yields the following quasilinear equation for F

$$F_t + (b - \rho\lambda a)\, F_y + \frac{1}{2}a^2 F_{yy} - \frac{1}{2}\rho^2 a^2 \frac{F_y^2}{F} = \frac{1}{2}\lambda^2 F.$$

We next substitute $F(t,y) = [v(t,y)]^\delta$ for some constant δ into the previous equation and find that v sloves

$$v_t + (b - \rho\lambda a)\, v_y + \frac{1}{2}a^2 v_{yy} + \left(\frac{(\delta-1)a^2}{2} - \frac{\delta\rho^2 a^2}{2}\right)\frac{v_y^2}{v} = \frac{1}{2\delta}\lambda^2 v,$$

which clearly can be linearized by picking $\delta = \frac{1}{1-\rho^2}$. This leads to the Cauchy problem

$$v_t + (b - \rho\lambda a)\, v_y + \frac{1}{2}a^2 v_{yy} = \frac{1}{2}(1-\rho^2)\lambda^2 v, \tag{A.12}$$

for $(t,y) \in [0,\tau]\times\mathbb{R}$, with terminal condition $v(\tau,y) = \exp\left\{-\frac{1}{2}(1-\rho^2)\int_\tau^T \lambda^2(s)ds\right\}\cdot \exp\left\{-\gamma(1-\rho^2)c(y)\right\}$. To this end, we conclude that the value function V^c can be written as

$$V^c(t,x,y) = -e^{-\gamma x}[v(t,y)]^{\frac{1}{1-\rho^2}}, \tag{A.13}$$

where v is a solution of (A.12).

Probabilistic Representation. By the Girsanov's theorem, we first observe that the process $\tilde{W}_t^1 = W_t^1 + \int_0^t \lambda(s)ds, t \geq 0$, is a standard Brownian motion under the martingale measure $\mathbb{Q}$ defined in (4.15). Further the dynamics of the nontraded asset Y can be written as

$$dY_s = (b(s, Y_s) - \rho\lambda a(s, Y_s))\, ds + a(s, Y_s)d\tilde{W}_s, \quad t \leq s,$$

with $Y_t = y \in \mathbb{R}$, where $\tilde{W}_t = \rho\tilde{W}_t^1 + \epsilon W_t^{\perp}, t \geq 0$, is a Brownian motion under measure $\mathbb{Q}$. Therefore, using Feynman-Kac connection, we conclude that the solution v to (A.12) can be represented by

$$v(t, y) = \exp\left\{-\frac{1}{2}(1-\rho^2)\int_t^T \lambda^2(s)ds\right\} E_{\mathbb{Q}}\left[\exp\left\{-\gamma(1-\rho^2)c(Y_\tau)\right\} \big| Y_t = y\right], \tag{A.14}$$

under the martingale measure $\mathbb{Q}$. Indeed, plugging $v = \exp\left\{-\frac{1}{2}(1-\rho^2)\int_t^T \lambda^2(s)ds\right\} f(t, y)$ into equation (A.12), we find that f solves

$$f_t + (b - \rho\lambda a)f_y + \frac{1}{2}a^2 f_{yy} = 0,$$

with terminal condition $f(\tau, y) = \exp\{-\gamma(1-\rho^2)c(y)\}$. Applying Itô's formula to $f(s, Y_s)$ and combining with the previous equation yields $df(s, Y_s) = af_y(s, Y_s)d\tilde{W}_s$, which shows that process $f(s, Y_s)$ is a local martingale. This leads to $f(t, y) = E[f(\tau, Y_\tau)|Y_t = y] = E[\exp\{-\gamma(1-\rho^2)c(y)\}|Y_t = y]$, and thus the representation (A.14) follows. Therefore, Proposition 4.2.3 follows by denoting

$$(\mathcal{E}_{\mathbb{Q}}^{(t,T)}c)(y) = -\frac{1}{\gamma(1-\rho^2)}\ln v(t, y).$$

Finally, according to the Verification Theorem, our candidate solution V is indeed the value function since it is smooth and the optimal policy is given as follows.

Construct the Optimal Policy. Combining (A.10), (A.13) and (A.14), we construct the optimal policy in feedback form $\Pi_s^* = \pi^*(s, X_s^*, Y_s)$, where the feedback function

$$\pi^*(t, x, y) = \frac{1}{\gamma}\frac{\lambda(t)}{\sigma(t)} - \rho\frac{a(t, y)}{\sigma(t)}\frac{d}{dy}(\mathcal{E}_{\mathbb{Q}}^{(t,T)}c)(y), \tag{A.15}$$

is independent of x, and the optimal wealth process X_s^* solves the SDE

$$dX_s^* = \mu(s)\Pi_s^* ds + \sigma(s)\Pi_s^* dW_s^1, \tag{A.16}$$

for $s \in [t, \tau]$, with initial condition $X_t^* = x > 0$. $\square$

A.3 Proofs of Chapter 5

Proof of Proposition 5.3.2

We first recall the HJB equation (5.12). Applying the first-order condition of the maximization yields the optimal control policy in feedback form

$$\pi^{0,*} = -\frac{\mu M_x + \sigma^2 S M_{Sx}}{\sigma^2 M_{xx}}. \tag{A.17}$$

By plugging the above feedback control into the HJB equation (5.12) we obtain

$$M_t - \frac{\left(\mu M_x + \sigma^2 S M_{Sx}\right)^2}{2\sigma^2 M_{xx}} + \mathcal{L}^{(S)} M = 0,$$

for $(t, S, x) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}$, with terminal condition $M(T, S, x) = -e^{-\gamma x}$. Substituting the ansatz transformation $M(t, S, x) = -\exp\{-\gamma(x + m(t, S))\}$ into the previous equation, we find that $m(t, S)$ solves

$$m_t + \frac{1}{2}\sigma^2 S^2 m_{SS} + \frac{1}{2\gamma}\frac{\mu^2}{\sigma^2} = 0, \tag{A.18}$$

for $(t, S) \in [0, T] \times \mathbb{R}^+$, with terminal condition $m(T, S) = 0$. Using the Feynman-Kac connection, we conclude that m can be represented as expectation (5.14) under martingale measure $\mathbb{Q}$ specified by (5.15). Indeed, the Girsanov's theorem implies that the process $\tilde{W}_t^1 := W_t^1 + \int_0^t \mu(s, S_s)/\sigma(s, S_s) ds, t \in [0, T]$, is a standard Brownian motion under measure $\mathbb{Q}$. Applying Itô's formula to $m(s, S_s)$ and combining equation (A.18) yields

$$0 = m(T, S_T) = m(t, S_t) - \frac{1}{2\gamma}\int_t^T \frac{\mu^2(s, S_s)}{\sigma^2(s, S_s)} ds + \int_t^T \sigma(s, S_s) S_s m_S(s, S_s) d\tilde{W}_s^1.$$

The representation (5.14) thus follows by taking conditional expectation under measure $\mathbb{Q}$ on both sides of the above equality. Furthermore, by substituting (5.13) into (A.17), we obtain the optimal policy $\Pi_s^{0,*}$ in feedback form $\pi_s^{0,*}(s, S_s, X_s^{0,*})$ given by

$$\pi^{0,*}(t,S,x) = \frac{1}{\gamma}\frac{\mu(t,S)}{\sigma^2(t,S)} - Sm_S(t,S),$$

for $(t,S,x) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$. The optimal wealth process is then given by

$$dX_s^{0,*} = \mu(s,S_s)\Pi_s^{0,*}ds + \sigma(s,S_s)\Pi_s^{0,*}dW_s^1,$$

for $s \in [t,T]$, with initial condition $X_t^{0,*} = x \in \mathbb{R}$. Finally, the classical verification results [see, for example, 50, Theorem IV.3.1] ensure the optimality of our solution. $\square$

Proof of Proposition 5.3.3

We first recall the HJB equation (5.18). The associated optimal control policy is given in feedback form

$$\pi^* = -\frac{\mu V_x + \rho\sigma a V_{xy} + \sigma^2 S V_{Sx}}{\sigma^2 V_{xx}}. \tag{A.19}$$

By plugging the above feedback control into equation (5.18), we rewrite the HJB equation as

$$V_t - \frac{\left(\mu V_x + \rho\sigma a V_{xy} + \sigma^2 S V_{Sx}\right)^2}{2\sigma^2 V_{xx}} + \mathcal{L}^{(S,y)}V = 0,$$

for $(t,S,x,y) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$, with terminal condition $V(T,S,x,y) = -\exp\{-\gamma(x+g(S,y))\}$. Thus, substituting the ansatz form $\tilde{V} = -\exp\{-\gamma(x+m(t,S)+h(t,S,y))\}$ into the previous equation and combining (A.18), we obtain the quasilinear equation

$$h_t + \tilde{\mathcal{L}}^{(S,y)}h - \frac{1}{2}\gamma(1-\rho^2)a^2h_y^2 = 0,$$

for $(t,S,y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$, with terminal condition $h(T,S,y) = g(S_T, Y_T)$, where $\tilde{\mathcal{L}}^{(S,y)}$ is given in (5.11). This is the familiar HJB equation studied in [49, section VI.5]. Under our assumption on the market coefficients, the solution h is bounded and unique in the class $C^{1,2,2}([0,T]\times\mathbb{R}^+\times\mathbb{R})$ [see also 116]. This in turn yields the same properties for $\tilde{V}$. Following the similar arguments as in the proof of Theorem 2.2 in [129], we identify $\tilde{V}$ as the unique viscosity solution of (5.18) and conclude $V^0=\tilde{V}$. We next substitute (5.19) into (A.19) to construct the optimal feedback policy

$$\pi^*(t,S,x,y)=\frac{1}{\gamma}\frac{\mu(t,S)}{\sigma^2(t,S)}-Sm_S(t,S)-Sh_S(t,S,y)-\rho\frac{a(t,S)}{\sigma(t,S)}h_y(t,S,y).$$

for $t\in[t_1,T]$. Finally, the optimality of the strategy follows from the classical verification results. $\square$

Bibliography

[1] Dong Hyun Ahn and Bin Gao. A parameteric nonlinear model of term structure dynamics. *Review of Financial Studies*, 12(4):721–762, 1999.

[2] Aurélien Alfonsi. On the discretization schemes for the cir (and bessel squared) processes. *Monte Carlo Methods and Applications*, 11(4):355–384, 2005.

[3] Franklin Allen and Anthony M. Santomero. The theory of financial intermediation. *Journal of Banking and Finance*, 21(11):1461–1485, 1998.

[4] Yakov Amihud and Haim Mendelson. Dealership market: Market-making with inventory. *Journal of Financial Economics*, 8:31–53, 1980.

[5] Kaushik Amin, Joshua D. Coval, and H. Nejat Seyhun. Index option prices and stock market momentum. *Journal of Business*, 77(4):835–874, 2004.

[6] Takuji Arai. The relations between minimal martingale measure and minimal entropy martingale measure. *Asia-Pacific Financial Markets*, 8:167–177, 2001.

[7] Gurdip S. Bakshi, Charles Q. Cao, and Zhiwu Chen. Empirical performance of alternative option pricing models. *Journal of Finance*, 52(6):2003–2049, 1997.

[8] Pauline Barrieu and Nicole El Karoui. Optimal design of derivatives in illiquid framework. *Quantitative Finance*, 2(1):1–8, 2002.

[9] David S. Bates. Post '87 crash fears in the S&P futures options market. *Journal of Econometrics*, 94(1):181–238, 2000.

[10] Dirk Becherer. Rational hedging and valuation of integrated risks under constant absolute risk aversion. *Insurance: Mathematics and Economics*, 33(1):1–28, 2003.

[11] Hatem Ben-Ameur, Michèle Breton, and Pascal François. A dynamic programming approach to price installment options. *European Journal of Operational Research*, 169(2):667–676, 2006.

[12] Fred E. Benth and Kenneth H. Karlsen. A PDE representation of the density of the minimal entropy martingale measure in stochastic volatility markets. *Stochastics and Stochastics Reports*, 77(2):109–137, 2005.

[13] Dirk Bergemann and Ulrich Hege. Venture capital financing, moral hazard and learning. *Journal of Banking and Finance*, 22(6-8):703–735, 1998.

[14] Fisher Black and Myron S. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[15] Nicolas P. B. Bollen and Robert E. Whaley. Does net buying pressure affect the shape of implied volatility functions? *Journal of Finance*, 59(2):711–753, 2004.

[16] Oleg Bondarenko. Why are put options so expensive? Working paper, University of Illinois at Chicago, November 2003.

[17] Hans Bühlmann. *Mathematical Methods in Risk Theory.* Springer-Verlag, 1970.

[18] Peter Carr, Hélyette Geman, Dilip B. Madan, and Marc Yor. Pricing options on realized variance. *Finance and Stochastics*, 9(4):453–475, 2005.

[19] Peter Carr, Xing Jin, and Dilip B. Madan. Optimal investment in derivative securities. *Finance and Stochasitcs*, 5(1):33–59, 2001.

[20] Peter Carr and Dilip B. Madan. Optimal positioning in derivative securities. *Quantitive Finance*, 1(1):19–37, 2001.

[21] Kam C. Chan, Louis T. W. Cheng, and Peter P. Lung. Net buying pressure, volatility smile, and abnormal profit of Hang Seng index options. *Journal of Futures Markets*, 24(12):1165–1194, 2004.

[22] Kam C. Chan, Louis T. W. Cheng, and Peter P. Lung. Testing the net buying pressure hypothesis during the Asian financial crisis: Evidence from Hang Seng index options. *Journal of Financial Research*, 29(1):43–62, 2006.

[23] Sébastien Chaumont, Peter Imkeller, and Matthias Müller. Equilibrium trading of climate and weather risk and numerical simulation in a Markovian framework. *Stochastic Environmental Research and Risk Assessment*, 20(3):184–205, 2006.

[24] Pierangelo Ciurlia and Ilir Roko. Valuation of American continuous-installment options. *Computational Economics*, 25(1-2):143–165, 2005.

[25] Stijn Claessens, Thomas Glaessner, and Daniela Klingebiel. Electronic finance: Reshaping the financial landscape around the world. *Journal of Financial Servies Research*, 22(1/2):29–61, 2002.

[26] George M. Constantinides, Jens C. Jackwerth, and Stylianos Perrakis. Mispricing of S&P 500 index options. Working paper, University of Chicago, May 2006.

[27] Didier Cossin, Benoît Leleux, and Entela Saliasi. Understanding the economic value of legal covenants in investment contracts: A real-options approach to venture equity contracts. FAME working paper, University of Lausanne, Switzerland, 2002.

[28] Joshua D. Coval and Tyler Shumway. Expected option returns. *Journal of Finance*, 56(3):983–1009, 2001.

[29] John C. Cox, Johnathan E. Ingersoll, and Stephen A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.

[30] Dwight B. Crane, Zvi Bodie, Kenneth A. Froot, Scott P. Mason, Robert C. Merton, and Andre F. Perold. *The Global Financial System: A Functional Perspective*. Harvard Business School Press, 1995.

[31] I. Csiszár. I-divergence geometry of probability distributions. *Annals of Probability*, 3:146–158, 1975.

[32] Mark H. A. Davis. Option pricing in incomplete markets. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of Derivative Securities*. Cambridge University Press, Cambridge, UK, 1997.

[33] Mark H. A. Davis. Option valuation and hedging with basis risk. In T. E. Djaferis and I. C. Schick, editors, *Systems Theory: Modelling, Analysis and Control*. Kluwer, Amsterdam, 1999.

[34] Mark H. A. Davis. Optimal hedging with basis risk. Preprint, Technical University of Vienna, 2000.

[35] Mark H. A. Davis, Vassilios G. Panas, and Thaleia Zariphopoulou. European option pricing with transaction costs. *SIAM Journal on Control and Optimization*, 31(2):470–493, 1993.

[36] Mark H. A. Davis, Walter Schachermayer, and Robert G. Tompkins. Pricing, no-arbitrage bounds and robust hedging of installment options. *Quantitative Finance*, 1:597–610, 2001.

[37] Mark H. A. Davis, Walter Schachermayer, and Robert G. Tompkins. Installment options and static hedging. *Journal of Risk Finance*, 3(2):46–52, 2002.

[38] Mark H. A. Davis, Walter Schachermayer, and Robert G. Tompkins. The evaluation of venture capital as an installment option: Valuing real options using real options. *Zeitschrift für Betriebswirtschaft*, 3:77–96, 2004.

[39] Mark H. A. Davis and Thaleia Zariphopoulou. American options and transaction fees. In Mark H. A. Davis, Darrell Duffie, Wendell H. Fleming, and Steven E. Shreve, editors, *Mathematical Finance*, volume 65 of *IMA Volumes in Mathematics and Its Applications*, pages 47–61. Springer-Verlag, 1995.

[40] Freddy Delbaen, Peter Grandits, Thorsten Rheinlander, Dominick Samperi, Martin Schweizer, and Christophe Stricker. Exponential hedging and entropic penalties. *Mathematical Finance*, 12(2):99–123, 2002.

[41] Freddy Delbaen and Walter Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.

[42] Freddy Delbaen and Walter Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen*, 312:215–250, 1998.

[43] Avinash K. Dixit and Robert S. Pindyck. *Investment under Uncertainty*. Princeton University Press, Princeton, New Jersey, 1994.

[44] Darrell Duffie, Wendell H. Flemming, H. Mete Soner, and Thaleia Zariphopoulou. Hedging in incomplete markets with HARA utility. *Journal of Economic Dynamics and Control*, 21(4-5):753–782, 1997.

[45] Darrell Duffie and Rohit Rahi. Financial market innovation and security design: An introduction. *Journal of Economic Theory*, 65(1):1–42, 1995.

[46] Darrell Duffie and Thaleia Zariphopoulou. Optimal investment with undiversifiable income risk. *Mathematical Finance*, 3(4-5):135–148, 1993.

[47] Eugene F. Fama. Banking in the theory of finance. *Journal of Monetary Economics*, 6(1):39–57, 1980.

[48] Stephen Figlewski. Options arbitrage in imperfect markets. *Journal of Finance*, 44(5):1289–1311, 1989.

[49] Wendell H. Fleming and Raymond W. Rishel. *Deterministic and Stochastic Optimal Control*. Springer-Verlag, New York, 1975.

[50] Wendell H. Fleming and H. Mete Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York, 1993.

[51] Lucien Foldes. Valuation and martingale properties of shadow prices: An exposition. *Journal of Economic Dynamics & Control*, 24:1641–1701, 2000.

[52] Hans Föllmer and Martin Schweizer. Hedging of contingent claims under incomplete information. In Mark H. A. Davis and Robert J. Elliott, editors, *Applied Stochastic Analysis*, volume 5 of *Stochastic Monographs*, pages 389–414. Gordon and Breach, New York, 1991.

[53] Marco Frittelli. Introduction to a theory of value coherent with the no-arbitrage principle. *Finance and Stochastics*, 4(3):275–297, 2000.

[54] Marco Frittelli. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance*, 10(1):39–52, 2000.

[55] Peter Friz and Jim Gatheral. Valuation of volatility derivatives as an inverse problem. *Quantative Finance*, 5(6):531–542, 2005.

[56] Nicolae Gârleanu, Lasse H. Pedersen, and Allen M. Poteshman. Demand-based option pricing. Working paper, The Wharton School of Business, August 2006.

[57] Mark B. Garman. Market microstructure. *Journal of Financial Economics*, 3:257–275, June 1976.

[58] Jim Gatheral. *The Volatility Surface: A Practitioner's Guide*. John Wiley & Sons, Inc., 2006.

[59] Hans U. Gerber. *An Introduction to Mathematical Risk Theory*. Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, 1979.

[60] Hans U. Gerber and Elias S.W. Shiu. Option pricing by esscher transforms. *Transactions of Society of Actuaries*, 46:99–140, 1994.

[61] Robert Geske. The valuation of corporate liabilities as compound options. *Journal of Finance and Quantitative Analysis*, 12(4):541–552, 1977.

[62] Robert Geske. The valuation of compound options. *Journal of Financial Economics*, 7(1):63–81, 1979.

[63] Paul A. Gompers. Optimal investment, monitoring, and the staging of venture capital. *Journal of Finance*, 50(5):1461–1489, 1995.

[64] Peter Grandits and Thorsten Rheinländer. On the minimal entropy martingale measure. *Annals of Probability*, 30(3):1003–1038, 2002.

[65] M. R. Grasselli and T. R. Hurd. Indifference pricing and hedging for volatility derivatives. Working paper, McMaster University, 2006.

[66] Nils Hakansson. The fantastic world of finance: Progress and the free lunch. *Journal of Financial and Quantative Analysis*, 14(4):717–734, June 1979.

[67] Larry Harris. *Trading & Echanges: Market Microstructure for Practitioners.* Financial Management Association Survey and Synthesis. Oxford University Press, New York, 2003.

[68] Vicky Henderson. Valuation of claims on nontraded assets using utility maximization. *Mathematical Finance*, 12(4):351–373, 2002.

[69] Steven L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(3):327–343, 1993.

[70] Thomas Ho and Hans R. Stoll. The dynamics of dearler markets under competition. *Journal of Finance*, 38(4):1053–1074, 1983.

[71] David Hobson. Real options, non-traded assets and utility indifference prices. Preprint, University of Bath, 2003.

[72] Stewart D. Hodges and Anthony Neuberger. Optimal replication of contingent claims under transaction costs. *Review of Futures Markets*, 8:222–239, 1989.

[73] Yang Hu, Peter Imkeller, and Matthias Müller. Partial equilibrium and market completion. *International Journal of Theoretical and Applied Finance*, 8(4):483–508, 2005.

[74] Julien D. Hugonnier, Dmitry Kramkov, and Walter Schachermayer. On utility based pricing of contingent claims in incomplete markets. *Mathematical Finance*, 15(2):203–212, 2005.

[75] Shunsuke Ihara. *Information Theory for Continuous Systems.* Series on Probability & Statistics. World Scientific, Singapore, 1993.

[76] Aytaç İlhan, Mattias Jonsson, and Ronnie Sircar. Portfolio optimization with derivatives and indifference pricing. Preprint, Princeton University, September 2004.

[77] Aytaç İlhan, Mattias Jonsson, and Ronnie Sircar. Optimal investment with derivative securities. *Finance and Stochastics*, 9(4):585–595, 2005.

[78] Yuri M. Kabanov and Christophe Stricker. On the optimal portfolio for the exponential utility maximization: Remarks to the six-author paper. *Mathematical Finance*, 12(2):125–134, 2002.

[79] Jan Kallsen. Utility-based derivative pricing in incomplete markets. In H. Geman, D. Madan, S.R. Pliska, and T. Vorst, editors, *Mathematical Finance - Bachelier Congress 2000*, pages 313–338. Springer, Berlin, 2001.

[80] Jan Kallsen. Derivative pricing based on local utility maximization. *Finance and Stochastics*, 6(1):115–140, 2002.

[81] Steven N. Kaplan and Per Strömberg. Financial contracting theory meets the real world: An empirical analysis of venture capital contracts. *Review of Economic Studies*, 70(2):281–315, 2003.

[82] Ioannis Karatzas, John P. Lehoczky, Steven E. Shreve, and Gan-Lin Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29:702–730, 1991.

[83] Dmitry Kramkov and Walter Schachermayer. The asymptotic elasticity of utility functions and optimal investments in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999.

[84] Dmitry Kramkov and Walter Schachermayer. Necessary and sufficient conditions in the problem of optimal investments in incomplete markets. *Annals of Applied Probability*, 13(4):1504–1516, 2003.

[85] Dmitry Kramkov and Mihai Sirbu. Asymptotic analysis of utility-based hedging strategies for small number of contingent claims. Working paper, Carnegie Mellon University, July 2006.

[86] Dmitry Kramkov and Mihai Sirbu. Sensitivity analysis of utility based prices and risk-tolerance wealth processes. Preprint, Carnegie Mellon University, May 2006.

[87] Damien Lamberton and Bernard Lapeyre. *An Introduction to Stochastic Calculus Applied to Finance*. Chapman and Hall/CRC, 1996.

[88] Beni Lauterbach and Paul Schultz. Pricing warrants: An empirical study of the Black-Scholes model and its alternatives. *Journal of Finance*, 45(4):1181–1209, 1990.

[89] Pierre-Louis Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. part 2: Viscosity solutions and uniqueness. *Communications in Partial Differential Equations*, 8:1299–1276, 1983.

[90] Jun Liu and Jun Pan. Dynamic derivative strategies. *Journal of Financial Economcis*, 69:401–430, 2003.

[91] Francis A. Longstaff. Option pricing and the martingale restriction. *Review of Financial Studies*, 8(4):1091–1124, 1995.

[92] Francis A. Longstaff and Eduardo S. Schwartz. Valuing American options by simulation: A simple least-squares approach. *Review of Financial Studies*, 14(1):113–147, 2001.

[93] Dilip Madan and Badih Soubra. Design and marketing of financial products. *The Reviews of Financial Studies*, 4(2):361–384, 1991.

[94] Ananth Madhavan and Seymour Smidt. An analysis of changes in specialist inventories and quotations. *Journal of Finance*, 48(5):1595–1628, 1993.

[95] Steven Manaster and Steven C. Mann. Life in the pits: Competitive market making and inventory control. *Review of Financial Studies*, 9(3):953–976, 1996.

[96] Robert C. Merton. Lifetime portfolio selection under uncertainty: The continuous case. *Review of Economics and Statistics*, 51(3):247–257, 1969.

[97] Robert C. Merton. Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4(1):141–183, 1973.

[98] Robert C. Merton. On the application of the continuous-time theory of finance to financial intemediation and insurance. *The Geneva Papers on Risk and Insurance*, 14(52):225–262, July 1989.

[99] Robert C. Merton. *Continuous-Time Finance*. Blackwell, Cambridge, MA, 1992.

[100] Robert C. Merton. Operation and regulation in financial intermediation: A functional prespective. In P. Englund, editor, *Operation and Regulation of Financial Markets*. The Ecomomic Council, Stockholm, 1993.

[101] Robert C. Merton. Financial innovation and the management and regulation of financial institutions. *Journal of Banking and Finance*, 19:461–481, 1995.

[102] Robert C. Merton. A functional perspective of financial intemeditation. *Financial Management*, 24(2):23–41, 1995.

[103] Robert C. Merton. Applications of option-pricing theory: 25 years later. *American Economic Review*, 88(3):323–349, 1998.

[104] Robert C. Merton. Future possibilities in finance theory and finance practice. In H. Geman, D. Madan, Pliska S., and T. Vorst, editors, *Mathematical Finance – Bachelier Congress 2000*. Springer-Verlag, Berlin, 2002.

[105] Robert C. Merton. Thoughts on the future: Theory and practice in investment management. *Financial Analysts Journal*, 59(1):17–23, 2003.

[106] Robert C. Merton and Zvi Bodie. Design of financial systems: Towards a synthesis of function and structure. *Journal of Investment Management*, 3(1):1–23, 2005.

[107] Michael Monoyios. Performance of utility-based strategies for hedging basis risk. *Quantitative Finance*, 4:245–255, 2004.

[108] Michael Monoyios. The minimal entropy measure and an esscher transform in an incomplete market model. Working paper, Mathematical Institute, University of Oxford, Oxford, UK, December 2005.

[109] Marek Musiela and Thaleia Zariphopoulou. Indifference prices and related measures. Technical report, University of Texas at Austin, 2001.

[110] Marek Musiela and Thaleia Zariphopoulou. Indifference prices of early exercise claims. In George Yin and Qing Zhang, editors, *Proceedings of an AMS-IMS-SIAM Joint Summer Research Conference on Mathematics of Finance*, volume 351 of *Contemporary Mathematics*, pages 259–271, 2003.

[111] Marek Musiela and Thaleia Zariphopoulou. An example of indifference prices under exponential preferences. *Finance and Stochastics*, 8:229–239, 2004.

[112] Marek Musiela and Thaleia Zariphopoulou. A valuation algorithm for indifference prices in incomplete markets. *Finance and Stochastics*, 8:399–414, 2004.

[113] Marek Musiela and Thaleia Zariphopoulou. Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model. Preprint, University of Texas at Austin, 2006.

[114] Marek Musiela and Thaleia Zariphopoulou. Optimal asset allocation under forward exponential performance criteria. Preprint, University of Texas at Austin, 2006.

[115] Mark P. Owen. Utility based optimal hedging in incomplete markets. *Annals of Applied Probability*, 12(2):691–709, 2002.

[116] Huyên Pham. Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints. *Applied Mathematics and Optimization*, 46(1):55–78, 2002.

[117] L.C.G. Rogers and David Williams. *Diffusions, Markov Processes and Martingales*. Cambridge University Press, Cambridge, UK, 2nd edition, 2000.

[118] Richard Rouge and Nicole El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.

[119] Mark Rubinstein. Double trouble. *RISK*, 5(1):73–76, 1991.

[120] William A. Sahlman. The structure and governance of venture capital organizations. *Journal of Financial Economics*, 27(2):473–521, 1990.

[121] Walter Schachermayer. Optimal investments when wealth may become negative. *Annals of Applied Probability*, 11(3):694–734, 2001.

[122] Walter Schachermayer. Introduction to the mathematics of financial markets. In P. Bernard, editor, *Lectures on Probability Theory and Statistics*, volume 1816 of *Lecture Notes in Mathematics*, pages 111–177. Springer, New York, 2003.

[123] Myron S. Scholes. Financial infrastructure and economic growth. In *Conference on Growth and Development: The Economics of the 21st Century*. Center for Economics Policy Research, Stanford University, June 1994.

[124] Bert Scholtens and Dick van Wensveen. The theory of financial intermediation: An essay on what it does (not) explain. Studies 1, SUERF, Vienna, 2003.

[125] Martin Schweizer. On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stochastic Analysis and Applications*, 13:573–599, 1995.

[126] Martin Schweizer. A minimality property of the minimal martingale measure. *Stochastics and Probability Letters*, 42:27–31, 1999.

[127] Michael J. P. Selby and Stewart D. Hodges. On the evaluation of coumpound options. *Management Science*, 33(3):347–355, 1987.

[128] Andrei Shleifer and Robert W. Vishny. The limits of arbitrage. *Journal of Finance*, 52(1):35–55, 1997.

[129] Ronnie Sircar and Thaleia Zariphopoulou. Bounds and asymptotic approximations for utility prices when volatility is random. *SIAM Journal on Control and Optimization*, 43(4):1328–1353, 2005.

[130] Seymour Smidt. Which road to an efficient stock market: Free competition or regulated monopoly? *Financial Analysts Journal*, 27(18-20):64–69, 1971.

[131] Sasha F. Stoikov. Pricing options from the point of view of a trader. Working paper, University of Texas at Austin, 2005.

[132] Sasha F. Stoikov and Thaleia Zariphopoulou. Optimal investment in the presence of unhedgeable risks and under CARA preferences. Preprint, IMA Volume in Mathematics and its Applications, 2005.

[133] Hans R. Stoll. The supply of dealer services in securities markets. *Journal of Finance*, 33(4):1133–1151, 1978.

[134] Hans R. Stoll. The stock exchange specialist system: An economic analysis. In *Monograph Series in Finance and Ecnomics*, volume 2, Salomon Brother Center, New York University, 1985.

[135] Uwe Wystup, Susanne Griebsch, and Christoph Kühn. FX installment options. Working paper, Goethe University, Frankfurt, 2004.

[136] Virginia R. Young. Pricing in an incomplete market with an affine term structure. *Mathematical Finance*, 14(3):359–381, 2004.

[137] Thaleia Zariphopoulou. A solution approach to valuation with unhedgeable risks. *Finance and Stochastics*, 5(1):61–82, 2001.

Vita

Qimou Su was born in Guangdong, China in 1979 to Jingqin and Bingqin Su. In 1997, he entered the South China University of Technology, Guangzhou, China, where he studied Mathematics and Computer Science. Within 3 years he got his degree of Bachelor of Science in Computer Science with minor in Mathematics. In 2002, he furthered his study at the Graduate School of the University of Texas at Austin, and joined the research group of Computational Finance at the Institute for Computational Engineering and Sciences (ICES). He received the degree of Master of Science in Computational and Applied Mathematics in May of 2004. He is expecting the degree of Doctor of Philosophy in Computational and Applied Mathematics with concentration on Quantitative Finance from The University of Texas at Austin in December of 2007.

Permanent Address: 1634 W 6TH ST APT B
Austin, TX 78703, USA

This dissertation was typeset with LaTeX 2_ε[1] by the author.

[1]LaTeX 2_ε is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The macros used in formatting this dissertation were written by Dinesh Das, Department of Computer Sciences, The University of Texas at Austin, and extended by Bert Kay, James A. Bednar, and Ayman El-Khashab.